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CBSE DRILL CLASS XI-XII

MATHEMATICS today

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for JEE (Main & Advanced)

$$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$$

$$\arccos(-x) = \pi - \arccos x \qquad \cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$$

$$\sin 2\alpha = 2\sin\alpha\times\cos\alpha$$

$$\arccos x = \pi - \arccos(-x) = \frac{\pi}{2} - \arcsin x = \operatorname{arcctg}\frac{x}{\sqrt{1-x^2}} \qquad \operatorname{arcctg}(-x) = \pi - \operatorname{arctg}x$$

$$\frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}} \qquad \operatorname{tg}\alpha\times\operatorname{ctg}\alpha = 1$$

$$\frac{\operatorname{ctg}^4\alpha - 6\operatorname{ctg}^2\alpha + 1}{4\operatorname{ctg}^3\alpha - 4\operatorname{ctg}\alpha}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

$$\operatorname{ctg}\alpha = \frac{\cos\alpha}{\sin\alpha}$$

$$-\frac{\cos\alpha}{\sin\alpha}\frac{\sin(\alpha+\beta)}{\sin\alpha\times\sin\beta}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos\alpha - \cos\beta$$

$$\frac{\operatorname{ctg}\alpha\times\operatorname{ctg}\beta \mp 1}{\operatorname{ctg}\beta \pm \operatorname{ctg}\alpha}$$

BRAIN @ WORK

MATHS MUSING
10 GREAT PROBLEMS

MATH ARCHIVES

CONCEPT MAP CLASS XI-XII

MOCK TEST PAPER
JEE MAIN

CHALLENGING PROBLEMS

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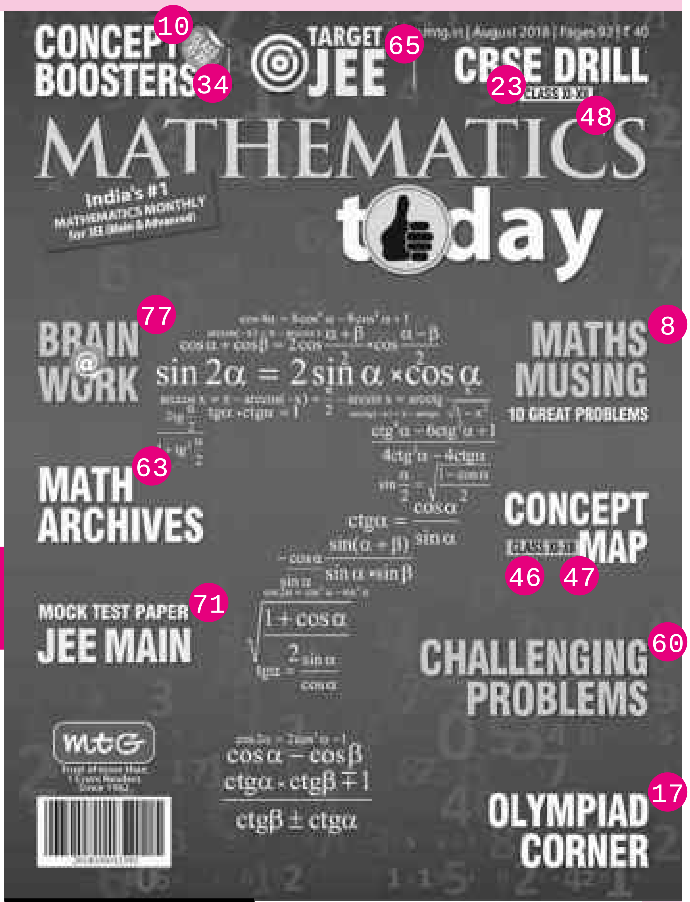

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 188

JEE MAIN

1. Five digit numbers are formed using the digits 0, 1, 2, 3, 4, 6, 7 without repetition. Find the probability that a number is divisible by 3.

(a) 600 (b) 700 (c) 800 (d) 900

2. If $p=\frac{1}{3}+\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}+\ldots\infty$, then

(a) $p^2-2p+2=0$ (b) $p^2+2p-2=0$

(c) $p^2-2p-2=0$ (d) none of these

3. The integral $\int\frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$ is equal to

(where C is a constant of integration.)

(a) $-2\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}+C$ (b) $-\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}+C$

(c) $-2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}+C$ (d) $2\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}+C$

4. If $f(x)=\frac{2^x-2^{-x}}{2^x+2^{-x}}$, then $f^{-1}(x)$ is

(a) $\frac{1}{2}\log_2\frac{x}{1-x}$ (b) $\frac{1}{2}\log_2\frac{1+x}{1-x}$

(c) $\frac{1}{2}\log_2\frac{1+x}{x}$ (d) $\frac{1}{2}\log_2\frac{2+x}{2-x}$

5. Equation of a line which is tangent to both the curves $y=x^2+1$ and $y=-x^2$ is

(a) $y=\sqrt{2}x-\frac{1}{2}$ (b) $y=\sqrt{2}x+\frac{1}{2}$

(c) $y=-\sqrt{2}x+\frac{1}{2}$ (d) $y=-\sqrt{2}x-\frac{1}{2}$

JEE ADVANCED

6. Let $\vec{x}$, $\vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\pi/3$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$ and $\vec{b}$ is non-zero vector perpendicular to $\vec{y}$ and $\vec{z}\times\vec{x}$, then

(a) $\vec{b}=(\vec{b}\cdot\vec{z})(\vec{z}-\vec{x})$ (b) $\vec{a}=(\vec{a}\cdot\vec{y})(\vec{y}-\vec{z})$

(c) $\vec{a}\cdot\vec{b}=-(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})$ (d) $\vec{a}=(\vec{a}\cdot\vec{y})(\vec{z}-\vec{y})$

COMPREHENSION

A variable straight line is drawn through the point $A(-1, 1)$ to intersect the parabola $y^2=4x$ at the points B and C. Let P be a point on the chord BC.

7. If AB, AP, AC are in H.P., then locus of P is

(a) a straight line (b) a pair of lines

(c) a circle (d) a parabola

8. If AB, AP, AC are in A.P., then the locus of P is

(a) a straight line (b) a pair of lines

(c) a circle (d) a parabola

INTEGER TYPE

9. If $I(x, y)$ is the incentre of the triangle formed by the points (3, 4), (4, 3), (1, 2), then x is

MATRIX MATCH

10. If $abc=1$ and $A=\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ is an orthogonal matrix.

Then, match the following columns.

	Column-I		Column-II
P.	The least value of $a+b+c$ is	1.	-1
Q.	The value of $ab+bc+ca$ is	2.	0
R.	The value of $a^2+b^2+c^2$ is	3.	1
S.	The value of $a^3+b^3+c^3$ can be	4.	2

	P	Q	R	S
(a)	3	2	1	4
(b)	4	3	2	1
(c)	1	2	3	4
(d)	1	3	2	4

See Solution Set of Maths Musing 187 on page no. 84

CONCEPT BOOSTERS

Sets and Relations

Class XI

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

SETS

A set is well defined class or collection of objects.

- **Roster method or Listing method :** In this method, a set is described by listing elements, separated by commas, within braces {}. The set of vowels of English alphabet may be described as $\{a, e, i, o, u\}$.
- **Set-builder method or Rule method :** In this method, a set is described by a characterizing property $P(x)$ of its elements x. In such a case, the set is described by $\{x : P(x)$ holds$\}$ or $\{x \mid P(x)$ holds$\}$, which is read as 'the set of all x such that $P(x)$ holds'.
  The set $A = \{0, 1, 4, 9, 16, ...\}$ can be written as $A = \{x^2 : x \in Z\}$.

TYPES OF SETS

- **Singleton set :** A set consisting of a single element is called a singleton set. The set {5} is a singleton set.
- **Null set or Empty set :** The set which contains no element at all is called the null set. This set is also called the 'empty set' or the 'void set'. It is denoted by the symbol ϕ or {}.
- **Finite set :** A set is called a finite set if it is either void set or its elements can be listed (counted) by a certain natural number.
  Cardinal number of a finite set : The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by $n(A)$ or $O(A)$.
- **Infinite set :** A set whose elements cannot be listed (counted) by a certain natural number (n) is called an infinite set.
- **Equivalent sets :** Two finite sets A and B are equivalent, if their cardinal numbers are same *i.e.* $n(A) = n(B)$.
  If $A = \{1, 3, 5, 7\}$; $B = \{10, 12, 14, 16\}$ then A and B are equivalent sets, as $O(A) = O(B) = 4$.
- **Equal sets :** Two sets A and B are said to be equal iff every element of A is an element of B and also every element of B is an element of A. Symbolically, $A = B$ if $x \in A \Leftrightarrow x \in B$.
  If $A = \{2, 3, 5, 6\}$ and $B = \{6, 5, 3, 2\}$, then $A = B$ because each element of A is an element of B and vice-versa.
  Note : Equal sets are always equivalent but equivalent sets need not to be equal sets.

SUBSETS (SET INCLUSION)

- Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.
  If A is subset of B, we write $A \subseteq B$, which is read as "A is a subset of B" or "A is contained in B".
  Thus, $A \subseteq B$ *i.e.*, $a \in A \Rightarrow a \in B$.
  The total number of subsets of a finite set containing n elements is 2^n.
- **Proper and improper subsets :** If A is a subset of B and $A \neq B$, then A is a proper subset of B. We write this as $A \subset B$.
  The null set ϕ is subset of every set and every set is subset of itself, *i.e.*, $\phi \subset A$ and $A \subseteq A$ for every set A. They are called improper subsets of A. It should be noted that ϕ has only one subset ϕ which is improper.

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He trains IIT and Olympiad aspirants.

All other subsets of A are called its proper subsets. Let $A = \{1, 2\}$. Then A has ϕ, $\{1\}$, $\{2\}$, $\{1, 2\}$ as its subsets out of which ϕ and $\{1, 2\}$ are improper and $\{1\}$ and $\{2\}$ are proper subsets.

UNIVERSAL SET

A set that contains all sets in a given context is called the universal set.
It should be noted that universal set is not unique.

POWER SET

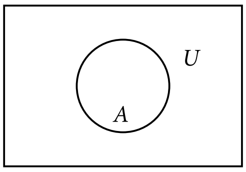
If S is any set, then the set of all the subsets of S is called the power set of S.
The power set of S is denoted by $P(S)$. Symbolically, $P(S) = \{T : T \subseteq S\}$. Obviously ϕ and S are both elements of $P(S)$.
Let $S = \{a, b, c\}$, then $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
Note: Power set of a given set is always non-empty.

DISJOINT SETS

Two sets A and B are said to be disjoint, if $A \cap B = \phi$. If $A \cap B \neq \phi$, then A and B are said to be non-intersecting or non-overlapping sets.

VENN DIAGRAMS

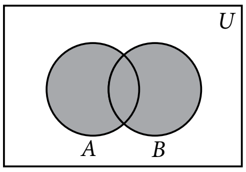
The combination of rectangles and circles are called Venn diagrams.
The universal set is usually represented by a rectangle and its subsets by circles.



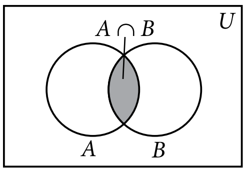
Note : If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. Two disjoints sets are represented by two non-intersecting circles.

OPERATIONS ON SETS

- **Union of sets :** Let A and B be two sets. The union of A and B is the set of all elements which are either in set A or in B. We denote the union of A and B by $A \cup B$ (read as "A union B"). Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$. Shaded portion in the given figure represents $A \cup B$.

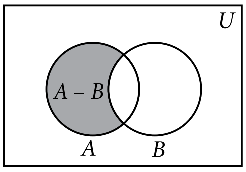


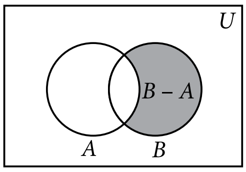
- **Intersection of sets :** Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.



The intersection of A and B is denoted by $A \cap B$ (read as "A intersection B").
Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

- **Difference of sets :** Let A and B be two sets. The difference of A and B written as $A - B$, is the set of all those elements of A which do not belong to B.



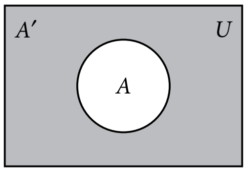


Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$
Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A
$\therefore \quad B - A = \{x : x \in B \text{ and } x \notin A\}$.
Note : For three sets A, B and C,
$(A - B) - C \neq A - (B - C)$.

- **Symmetric difference of two sets :** Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.
Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$.

COMPLEMENT OF A SET

Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $U - A$ and is defined the set of all those elements of U which are not in A. Thus, $A' = \{x \in U : x \notin A\}$.



LAWS OF ALGEBRA OF SETS

- **Idempotent laws :** For any set A, we have
(i) $A \cup A = A$
(ii) $A \cap A = A$
- **Identity laws :** For any set A, we have
(i) $A \cup \phi = A$
(ii) $A \cap U = A$
i.e., ϕ and U are identity elements for union and intersection respectively.
- **Commutative laws :** For any two sets A and B, we have
(i) $A \cup B = B \cup A$
(ii) $A \cap B = B \cap A$
(iii) $A \Delta B = B \Delta A$
- **Associative laws :** If A, B and C are any three sets, then
(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$
(iii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
i.e., union, intersection and symmetric difference of two sets are associative.

- **Distributive laws :** If A, B and C are any three sets, then
  (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  i.e., union and intersection are distributive over intersection and union respectively.
- **De-Morgan's law :** If A, B and C are any three sets, then
  (i) $(A \cup B)' = A' \cap B'$
  (ii) $(A \cap B)' = A' \cup B'$
  (iii) $A - (B \cap C) = (A - B) \cup (A - C)$
  (iv) $A - (B \cup C) = (A - B) \cap (A - C)$
- If A and B are any two sets, then
  (i) $A - B = A \cap B'$;
  (ii) $B - A = B \cap A'$
  (iii) $A - B = A \Leftrightarrow A \cap B = \phi$
  (iv) $(A - B) \cup B = A \cup B$
  (v) $(A - B) \cap B = \phi$
  (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$
  (vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
- If A, B and C are any three sets, then
  (i) $A \cap (B - C) = (A \cap B) - (A \cap C)$
  (ii) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

SOME IMPORTANT RESULTS

If A, B and C are finite sets and U be the finite universal set, then

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.
- $n(A - B) = n(A) - n(A \cap B)$
  i.e., $n(A - B) + n(A \cap B) = n(A)$
- $n(A \Delta B)$ = Number of elements which belong to exactly one of A or $B = n((A - B) \cup (B - A))$
  $= n(A - B) + n(B - A)$
  [$\because (A - B)$ and $(B - A)$ are disjoint sets]
  $= n(A) - n(A \cap B) + n(B) - n(A \cap B)$
  $= n(A) + n(B) - 2n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
- Number of elements in exactly two of the sets A, B, C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
- Number of elements in exactly one of the sets $A, B, C = n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
- $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
- $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

CARTESIAN PRODUCT OF SETS

Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.
Thus, $A \times B = [(a, b) : a \in A \text{ and } b \in B]$
If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.
Let $A = \{a, b, c\}$ and $B = \{p, q\}$.
Then, $A \times B = \{(a, p), (a, q), (b, p), (b, q), (c, p), (c, q)\}$
Also, $B \times A = \{(p, a), (p, b), (p, c), (q, a), (q, b), (q, c)\}$
Note :

- Cartesian product of two sets is not commutative (in general).
- Let A and B two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

IMPORTANT THEOREMS ON CARTESIAN PRODUCT OF SETS

For any three sets A, B and C

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B - C) = (A \times B) - (A \times C)$
- $(A \times B) \times C \neq A \times (B \times C)$
- $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
- $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

If A and B are any two non-empty sets, then
$A \times B = B \times A \Leftrightarrow A = B$

- If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
- If $A \subseteq B$, then $A \times C \subseteq B \times C$, for any set C.
- If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$
- For any sets A, B, C and D
  $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

RELATIONS

Let A and B be two non-empty sets, then every subset of $A \times B$ defines a relation from A to B *i.e.*, every relation from A to B is a subset of $A \times B$.
Let $R \subseteq A \times B$ and $(a, b) \in R$. Then we say that a is related to b by the relation R and write it as $a \, R \, b$.
Total number of relations : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines a relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations, the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

DOMAIN AND RANGE OF A RELATION

Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \in R\}$

Range $(R) = \{b : (a, b) \in R\}$.

Note :

- The whole set B is called co-domain of R.
- Range $\subseteq$ Co-domain

PROBLEMS

Single Correct Answer Type

1. Which of the following is the empty set ?
(a) $\{x : x$ is a real number and $x^2 - 1 = 0\}$
(b) $\{x : x$ is a real number and $x^2 + 1 = 0\}$
(c) $\{x : x$ is a real number and $x^2 - 9 = 0\}$
(d) $\{x : x$ is a real number and $x^2 = x + 2\}$.

2. If a set A has n elements, then the total number of subsets of A is
(a) n (b) n^2 (c) 2^n (d) $2n$

3. In a town of 10,000 families, it was found that 40% families buy newspaper X, 20% buy newspaper Y and 10% families buy newspaper Z, 5% families buy both X and Y, 3% buy both Y and Z and 4% buy both X and Z. If 2% families buy all the three newspapers, then number of families which buy newspaper X only is
(a) 3100 (b) 3300 (c) 2900 (d) 1400

4. In a city, 20 percent of the population travels by car, 50 percent travels by bus and 10 percent travels by both car and bus. Then persons travelling by car or bus is
(a) 80 percent (b) 40 percent
(c) 60 percent (d) 70 percent

5. In a class of 55 students, the number of students studying different subjects are 23 in Mathematics, 24 in Physics, 19 in Chemistry, 12 in Mathematics and Physics, 9 in Mathematics and Chemistry, 7 in Physics and Chemistry and 4 in all the three subjects. The number of students who have taken exactly one subject is
(a) 18 (b) 20
(c) 12 (d) none of these

6. If A, B and C are any three sets, then $A \times (B \cup C)$ is equal to
(a) $(A \times B) \cup (A \times C)$ (b) $(A \cup B) \times (A \cup C)$
(c) $(A \times B) \cap (A \times C)$ (d) none of these

7. If $A = \{2, 4, 5\}$, $B = \{7, 8, 9\}$ then $n(A \times B)$ is equal to
(a) 6 (b) 9 (c) 3 (d) 0

8. The smallest set A such that $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$ is
(a) $\{2, 3, 5\}$ (b) $\{3, 5, 9\}$
(c) $\{1, 2, 5, 9\}$ (d) none of these

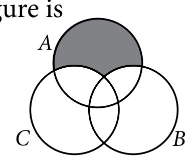
9. If A and B are two sets, then $A \cup B = A \cap B$ iff
(a) $A \subseteq B$ (b) $B \subseteq A$
(c) $A = B$ (d) none of these

10. If $A = \{2, 3, 4, 8, 10\}$, $B = \{3, 4, 5, 10, 12\}$, $C = \{4, 5, 6, 12, 14\}$, then $(A \cap B) \cup (A \cap C)$ is equal to
(a) $\{3, 4, 10\}$ (b) $\{2, 8, 10\}$
(c) $\{4, 5, 6\}$ (d) $\{3, 5, 14\}$

11. If A and B are two sets, then $A \cap (A \cup B)'$ is equal to
(a) A (b) B
(c) ϕ (d) none of these

12. If $N_a = [an : n \in N]$, then $N_5 \cap N_7 =$
(a) N_7 (b) N_{10} (c) N_{35} (d) N_5

13. The shaded region in the given figure is
(a) $A \cap (B \cup C)$
(b) $A \cup (B \cap C)$
(c) $A \cap (B - C)$
(d) $A - (B \cup C)$



14. Let U be the universal set and $A \cup B \cup C = U$. Then $\{(A - B) \cup (B - C) \cup (C - A)\}'$ is equal to
(a) $A \cup B \cup C$ (b) $A \cup (B \cap C)$
(c) $A \cap B \cap C$ (d) $A \cap (B \cup C)$

15. In a battle, 70% of the combatants lost one eye, 80% an ear, 75% an arm, 85% a leg, x% lost all the four limbs. The minimum value of x is
(a) 10 (b) 12
(c) 15 (d) none of these

16. Out of 800 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball. Of the total, 64 played both basketball and hockey; 80 played cricket and basketball and 40 played cricket and hockey; 24 played all the three games. The number of boys who did not play any game is
(a) 128 (b) 216 (c) 240 (d) 160

17. In a class of 100 students, 55 students have passed in Mathematics and 67 students have passed in Physics. Then the number of students who have passed in Physics only is
(a) 22 (b) 33 (c) 10 (d) 45

18. If A, B and C are any three sets, then $A - (B \cap C)$ is equal to
(a) $(A - B) \cup (A - C)$ (b) $(A - B) \cap (A - C)$
(c) $(A - B) \cup C$ (d) $(A - B) \cap C$

19. If A, B, C are three sets, then $A \cap (B \cup C)$ is equal to
(a) $(A \cup B) \cap (A \cup C)$ (b) $(A \cap B) \cup (A \cap C)$
(c) $(A \cup B) \cup (A \cup C)$ (d) none of these

20. In a class of 30 pupils, 12 take English, 16 take physics and 18 take history. If all the 30 pupils take at least one subject and no one takes all three then the number of pupils taking exactly 2 subjects is
(a) 16 (b) 6 (c) 8 (d) 20

21. A class has 175 students. The following data shows the number of students obtaining one or more subjects. Mathematics–100, Physics–70, Chemistry–40; Mathematics and Physics–30, Mathematics and Chemistry–28, Physics and Chemistry –23, Mathematics, Physics and Chemistry–18. How many students have offered Mathematics alone?
(a) 35 (b) 48 (c) 60 (d) 22

22. Given $n(U) = 20$, $n(A) = 12$, $n(B) = 9$, $n(A \cap B) = 4$, where U is the universal set, A and B are subsets of U, then $n((A \cup B)^c) =$
(a) 17 (b) 9 (c) 11 (d) 3

23. The relation R defined on the set of natural numbers as $\{(a, b) : a$ differs b by $3\}$, is given by
(a) $\{(1, 4), (2, 5), (3, 6),.....\}$
(b) $\{(4, 1), (5, 2), (6, 3),.....\}$
(c) Both (a) and (b) (d) none of these

24. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is
(a) 2^{mn} (b) $2^{mn} - 1$ (c) $2mn$ (d) m^n

25. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by
(a) $\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
(b) $\{(2, 2), (3, 2), (4, 2), (2, 4)\}$
(c) $\{(3, 3), (3, 4), (5, 4), (4, 3), (3, 1)\}$
(d) none of these

26. If A is the set of even natural numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from A to B is
(a) 2^9 (b) 9^2 (c) 3^2 (d) 2^{9-1}

27. The power set of $A = \{\phi, \{\phi\}\}$
(a) $\{\phi, \{\phi\}, \{\{\phi\}\}\}$
(b) $\{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$
(c) $\{\{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$
(d) None of these

28. $(A \cup B) \cap (A \cup B^C)$ equals
(a) A (b) B (c) $A \cap B'$ (d) $A \cup B'$

29. In a survey of 100 persons, it was found that 28 read magazine A, 30 read magazine B, 42 read magazine C, 8 read magazines A and B, 10 read magazines A and C, 5 read magazines B and C and 3 read all the three magazines. Then, number of persons who read none of the three magazines and magazine C only respectively are
(a) 20, 30 (b) 30, 20 (c) 25, 35 (d) 25, 40

30. Which of the following statements is true?
(a) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
(b) $A \subseteq B \subseteq C \Rightarrow A \subseteq C$
(c) $A \subseteq \phi \Rightarrow A = \phi$ (d) all of these

SOLUTIONS

1. (b) : Since, $x^2 + 1 = 0$, gives $x^2 = -1 \Rightarrow x = \pm i$
$\therefore$ x is not real but x is real (given)
$\therefore$ No value of x is possible.

2. (c) : Number of subsets of A having n elements $= 2^n$

3. (b) : Let sets A, B and C represents families who buy newspaper X, Y and Z respectively.
$n(A) = 40\%$ of $10000 = 4000$
$n(B) = 20\%$ of $10000 = 2000$
$n(C) = 10\%$ of $10000 = 1000$
$n(A \cap B) = 5\%$ of $10000 = 500$
$n(B \cap C) = 3\%$ of $10000 = 300$
$n(C \cap A) = 4\%$ of $10000 = 400$
$n(A \cap B \cap C) = 2\%$ of $10000 = 200$
We want to find $n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$
$= n(A) - n[A \cap (B \cup C)]$
$= n(A) - n[(A \cap B) \cup (A \cap C)]$
$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$
$= 4000 - [500 + 400 - 200] = 4000 - 700 = 3300.$

4. (c) : Let B and C represents the set of population who travel by bus and car respectively.
Given, $n(C) = 20$, $n(B) = 50$, $n(C \cap B) = 10$
Now, $n(C \cup B) = n(C) + n(B) - n(C \cap B)$
$= 20 + 50 - 10 = 60.$
Hence, required number of persons = 60%.

5. (d) : $n(M) = 23$, $n(P) = 24$, $n(C) = 19$
$n(M \cap P) = 12$, $n(M \cap C) = 9$, $n(P \cap C) = 7$
$n(M \cap P \cap C) = 4$
We have to find $n(M \cap P' \cap C')$, $n(P \cap M' \cap C')$, $n(C \cap M' \cap P')$
Now, $n(M \cap P' \cap C') = n[M \cap (P \cup C)']$
$= n(M) - n(M \cap (P \cup C))$
$= n(M) - n[(M \cap P) \cup (M \cap C)]$
$= n(M) - n(M \cap P) - n(M \cap C) + n(M \cap P \cap C)$
$= 23 - 12 - 9 + 4 = 27 - 21 = 6$
$n(P \cap M' \cap C') = n[P \cap (M \cup C)']$
$= n(P) - n[P \cap (M \cup C)]$
$= n(P) - n[(P \cap M) \cup (P \cap C)]$

$= n(P) - n(P \cap M) - n(P \cap C) + n(P \cap M \cap C)$
$= 24 - 12 - 7 + 4 = 9$
Similarly, $n(C \cap M' \cap P') = n(C) - n(C \cap P)$
$- n(C \cap M) + n(C \cap P \cap M)$
$= 19 - 7 - 9 + 4 = 23 - 16 = 7.$
Thus, the number of students who have taken exactly one subject $= 6 + 9 + 7 = 22$

6. (a)

7. (b) : $A \times B = \{(2, 7), (2, 8), (2, 9), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9)\}$
$\therefore \quad n(A \times B) = n(A) \cdot n(B) = 3 \times 3 = 9.$

8. (b) : Given $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$.
Hence, $A = \{3, 5, 9\}$.

9. (c) : Let $x \in A \Rightarrow x \in A \cup B$ [$\because A \subseteq A \cup B$]
$\Rightarrow \quad x \in A \cap B$ [$\because A \cup B = A \cap B$]
$\Rightarrow \quad x \in A$ and $x \in B \;\Rightarrow\; x \in B \Rightarrow A \subseteq B$
Similarly, $x \in B \;\Rightarrow\; x \in A \Rightarrow B \subseteq A$
Now, $A \subseteq B, B \subseteq A \quad \Rightarrow \quad A = B$

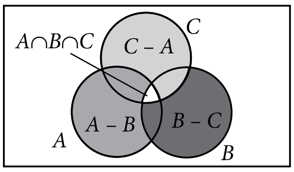
10. (a) : $A \cap B = \{2, 3, 4, 8, 10\} \cap \{3, 4, 5, 10, 12\}$
$= \{3, 4, 10\}$
Also, $A \cap C = \{4\}$
$\therefore \quad (A \cap B) \cup (A \cap C) = \{3, 4, 10\}.$

11. (c) : $A \cap (A \cup B)' = A \cap (A' \cap B')$
(De-Morgan's law)
$= (A \cap A') \cap B'$ (By associative law)
$= \phi \cap B' = \phi$

12. (c) : $N_5 \cap N_7 = N_{35}$,
[$\because$ 5 and 7 are relatively prime numbers].

13. (d)

14. (c) : From Venn diagram,
it is clear that,
$\{(A - B) \cup (B - C)$
$\cup (C - A)\}'$
$= A \cap B \cap C$.



15. (a) : Minimum value of
$n = 100 - (30 + 20 + 25 + 15) = 100 - 90 = 10$

16. (d) : Given, $n(C) = 224$, $n(H) = 240$, $n(B) = 336$
$n(H \cap B) = 64$, $n(B \cap C) = 80$
$n(H \cap C) = 40$, $n(C \cap H \cap B) = 24$
Number of boys who did not play any game
$= n(C^c \cap H^c \cap B^c) = n[(C \cup H \cup B)^c]$
$= \; n(U) - n(C \cup H \cup B)$
$= 800 - [n(C) + n(H) + n(B) - n(H \cap C)$
$-n(H \cap B) - n(C \cap B) + n(C \cap H \cap B)]$
$= 800 - [224 + 240 + 336 - 64 - 80 - 40 + 24] = 160.$

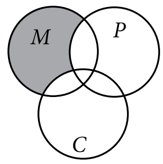
17. (d) : $n(M) = 55$, $n(P) = 67$, $n(M \cup P) = 100$
Now, $n(M \cup P) = n(M) + n(P) - n(M \cap P)$
$100 = 55 + 67 - n(M \cap P)$
$\therefore \quad n(M \cap P) = 122 - 100 = 22$
Now, $n(P \text{ only}) = n(P) - n(M \cap P) = 67 - 22 = 45.$

18. (a) : From De-morgan's law, we have
$A - (B \cap C) = (A - B) \cup (A - C).$

19. (b) : From distributive law, we have
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

20. (a) : Given, $n(E) = 12$, $n(P) = 16$, $n(H) = 18$
and $n(E \cup P \cup H) = 30$
$\because n(E \cup P \cup H) = n(E) + n(P) + n(H)$
$- n(E \cap P) - n(P \cap H) - n(E \cap H) + n(E \cap P \cap H)$
$\therefore \quad n(E \cap P) + n(P \cap H) + n(E \cap H) = 16$
Now, number of pupils taking exactly two subjects
$= n(E \cap P) + n(P \cap H) + n(E \cap H)$
$- 3n(E \cap P \cap H) = 16 - 0 = 16$

21. (c) : $n(M$ alone)
$= n(M) - n(M \cap C) - n(M \cap P)$
$+ n(M \cap P \cap C)$
$= 100 - 28 - 30 + 18$
$= 60.$



22. (d) : $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$= 12 + 9 - 4 = 17$
Now, $n((A \cup B)^c) = n(U) - n(A \cup B) = 20 - 17 = 3.$

23. (c) : $R = \{(a, b) : a, b \in N, |a - b| = 3\}$
$= \{(4, 1), (5, 2), (6, 3), ...\}$ or $\{(1, 4), (2, 5), (3, 6) ...\}.$

24. (a) : Number of relations from set A to set B having m and n elements respectively is 2^{mn}.

25. (d) : Here $R = \{(x, y) : |x^2 - y^2| < 16\}$
and given $A = \{1, 2, 3, 4, 5\}$
$\therefore \quad R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$

26. (a) : Here, $A = \{2, 4, 6\}$; $B = \{2, 3, 5\}$
$\therefore \quad A \times B$ contains $3 \times 3 = 9$ elements.
Hence, number of relations from A to $B = 2^9$.

27. (b) : $P(A) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

28. (a)

29. (a) : $n(A) = 28$, $n(B) = 30$, $n(C) = 42$, $n(A \cap B) = 8$, $n(A \cap C) = 10$, $n(B \cap C) = 5$, $n(A \cap B \cap C) = 3$
Since, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - (n(A \cap B)$
$+ n(A \cap C) + n(B \cap C)) + n(A \cap B \cap C)$
$= 100 - 23 + 3 = 80$
$\therefore \quad$ Number of persons who read none of the three magazines $= 100 - 80 = 20$
Also, n(read magazine C only)
$= n(C) - (n(A \cap C) + n(B \cap C)) + n(A \cap B \cap C)$
$= 42 - 10 - 5 + 3 = 30$

30. (d)

OLYMPIAD CORNER

1. Two unequal circles of radii R and r touch externally, and P and Q are the points of contact of a common tangent to the circles, respectively. Find the volume of the frustum of a cone generated by rotating PQ about the line joining the centres of the circles.

2. Let $n \geq 2$ be a natural number. Show that there exists a constant $C = C(n)$ such that for all real

   $x_1,..., x_n \geq 0$ we have $\sum_{k=1}^{n} \sqrt{x_k} \leq \sqrt{\prod_{k=1}^{n} (x_k + C)}$.

   Determine the minimum $C(n)$ for some values of n.

3. Find all real coefficients polynomials $p(x)$ satisfying $(x-1)^2 p(x) = (x-3)^2 p(x+2)$ for all x.

4. Prove that : $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$,

   where x, y, z are non-negative real numbers for which $x + y + z = 1$.

5. Find the value of the continued root:

   $$\sqrt{4+27\sqrt{4+29\sqrt{4+31\sqrt{4+33\sqrt{\dots}}}}}$$

6. A hexagon is inscribed in a circle with radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation $2r^3 - 7r - 3 = 0$.

7. Let $k \geq 2$ be an integer. The sequence (x_n) is defined

   by $x_0 = x_1 = 1$ and $x_{n+1} = \frac{x_n^k + 1}{x_{n-1}}$ for $n \geq 1$.

   (a) Prove that for each positive integer $k \geq 2$ the sequence (x_n) is a sequence of integers.

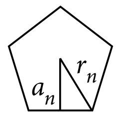
   (b) If $k = 2$, show that $x_{n+1} = 3x_n - x_{n-1}$ for $n \geq 1$.

8. If for every positive integer n, $f(n)$ is defined as

   $$f(n) = \begin{cases} 1 & \text{for } n = 1 \\ \dfrac{n}{f(n-1)} & \text{for } n \geq 2 \end{cases}$$

   then prove that : $\sqrt{1992} < f(1992) < \frac{4}{3}\sqrt{1992}$.

9. We consider regular n-gons with a fixed circumference 4. We call the distance from the centre of such a n-gon to a vertex r_n and the distance from the centre to an edge a_n.

   (a) Determine a_4, r_4, a_8, r_8

   (b) Give an appropriate interpretation for a_2 and r_2.

   (c) Prove: $a_{2n} = \frac{1}{2}(a_n + r_n)$ and $r_{2n} = \sqrt{a_{2n} r_n}$.

   (d) Let u_0, u_1, u_2, u_3, … be defined as follows :

   $u_0 = 0$, $u_1 = 1$, $u_n = \frac{1}{2}(u_{n-2} + u_{n-1})$ for even n

   and $u_n = \sqrt{u_{n-2}\, u_{n-1}}$ for odd n.

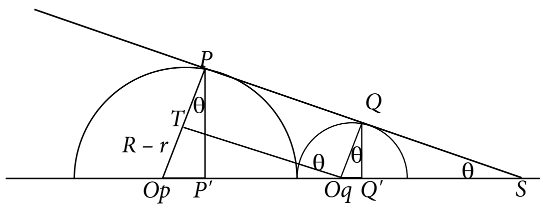
   Determine: $\lim_{n \to \infty} u_n$

10. There are real numbers a, b, c such that $a \geq b \geq c > 0$.

    Prove that $\frac{a^2 - b^2}{c} + \frac{c^2 - b^2}{a} + \frac{a^2 - c^2}{b} \geq 3a - 4b + c$.

SOLUTIONS

1.



First, we note that triangle $\Delta TOpOq$ has a right angle

angle at T, with $OpOq = R + r$ and $OpT = R - r$. Hence $OqT = 2\sqrt{Rr}$.

Because of parallel and perpendicular lines, all of angles $\angle PSOp$, $\angle TOqOp$, $\angle OpPP'$ and $\angle OqQQ'$ are equal. We denote the common value by θ. From triangle $\Delta OpOqT$, we note that

$$\sin\theta = \frac{R-r}{R+r} \text{ and } \cos\theta = \frac{2\sqrt{Rr}}{R+r}$$

Using various right triangles, we obtain:

$$OpP' = R\sin\theta = \frac{R(R-r)}{R+r},\ PP' = R\cos\theta = \frac{2R\sqrt{Rr}}{R+r}$$

$$P'S = PP'\cot\theta = \frac{2R\sqrt{Rr}}{R+r}\frac{2\sqrt{Rr}}{R-r} = \frac{4rR^2}{R^2-r^2},$$

$$QqQ' = r\sin\theta = \frac{r(R-r)}{R+r}$$

$$QQ' = r\cos\theta = \frac{2r\sqrt{Rr}}{R+r}$$

$$Q'S = QQ'\cot\theta = \frac{2r\sqrt{Rr}}{R+r}\frac{2\sqrt{Rr}}{R-r} = \frac{4r^2R}{R^2-r^2}$$

Hence, $V = \frac{\pi}{3}\left[(PP')^2P'S-(QQ')^2Q'S\right]$

$$= \frac{\pi}{3}\left[\frac{4R^3r}{(R+r)^2}\frac{4rR^2}{R^2-r^2} - \frac{4Rr^3}{(R+r)^2}\frac{4r^2R}{R^2-r^2}\right]$$

$$= \frac{16\pi R^2r^2(R^3-r^3)}{3(R+r)^2(R^2-r^2)} = \frac{16\pi R^2r^2(R^2+Rr+r^2)}{3(R+r)^3}$$

Which is the required volume.

2. We show that the inequality is valid for an aggregate of values of C of which the least is

$$C(n) = \frac{n-1}{\sqrt[n-1]{n^{n-2}}}, n \ge 2.$$

Let us first do the easier task of proving the existence of C's which make the inequality valid. Of course this part will be redundant as soon as we improve the technique to find the least C.

Setting $x_i = y_i^2$ where $y_i \ge 0$ ($i = 1,\ldots,n$), we are to show, equivalently, that for some C we have

$$\left(\sum_{i=1}^n y_i\right)^2 \le \prod_{i=1}^n y_i^2 + C \qquad \text{...(i)}$$

Treating the right hand side of (1) as a polynomial in C, we observe that all coefficients are non-negative and that the coefficient of C^{n-1} is Σy_i^2.

Thus, $\prod_{i=1}^n y_i^2 + C \ge \left(\sum_{i=1}^n y_i^2\right)C^{n-1}$

But by the Cauchy–Schwarz inequality, we have

$$\left(\sum_{i=1}^n y_i\right)^2 \le n\left(\sum_{i=1}^n y_i^2\right),$$

So inequality (i) will be valid if we choose $C = n^1/(n-1)$ or larger. This completes the easier task.

It turns out that $n^{1/(n-1)}$ is only a slight overestimate of the minimum C, which we now seek. for any C for which (i) is valid, set $w_i = yi\sqrt{n-1}/\sqrt{C}$, so that (i) becomes

$$\left(\sum_{i=1}^n w_i\right)^2 \le \frac{C^{n-1}}{(n-1)^{n-1}}\prod_{i=1}^n (w_i^2+n-1)$$

or equivalently

$$\left(\sum_{i=1}^n w_i\right)^2 \le \frac{C^{n-1}n^n}{(n-1)^{n-1}}\prod_{i=1}^n\left(\frac{w_i^2-1}{n}+1\right) \qquad \text{...(ii)}$$

To find the minimum C we shall first show that the following inequality is valid:

$$\left(\sum_{i=1}^n w_i\right)^2 \le n^2\prod_{i=1}^n\left(\frac{w_i^2-1}{n}+1\right) \qquad \text{...(iii)}$$

we shall use the weierstrass inequality

$$\prod_{i=1}^m (1+a_i) \ge 1 + \sum_{i=1}^m a_i$$

Which holds if all $a_i \le 0$ or if $-1 < a_i < 0$ for all i. Without loss of generality let $w_1,\ldots w_t \ge 1$ and $0 \le w_{t+1}, \ldots, w_n < 1$, where $t \in \{0, 1,\ldots,n\}$. Then

$$\prod_{i=1}^n\left(\frac{w_i^2-1}{n}+1\right) = \prod_{i=1}^n\left(\frac{w_i^2-1}{n}+1\right)\prod_{i=t+1}^n\left(\frac{w_i^2-1}{n}+1\right)$$

$$\ge \left(1+\sum_{i=1}^t\frac{w_i^2-1}{n}\right)\left(1+\sum_{i=t+1}^n\frac{w_i^2-1}{n}\right)$$

$$= \frac{1}{n^2}\left(n-t+\sum_{i=1}^t w_i^2\right)\left(t+\sum_{i=t+1}^n w_i^2\right)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^t w_i^2 + \sum_{i=t+1}^n 1^2\right)\left(\sum_{i=1}^t 1^2 + \sum_{i=t+1}^n w_i^2\right) \ge \frac{1}{n^2}\left(\sum_{i=1}^n w_i\right)$$

(the last inequality by the Cauchy-Schwarz inequality), which proves (iii). Note that equality occurs for $w_1 = \ldots w_n = 1$. We conclude that (ii) is valid for any C with $C^{n-1}n^n/(n-1)^{n-1} \ge n^2$, *i.e.*, with

$$C \ge \frac{n-1}{\sqrt[n-1]{n^{n-2}}}, n \ge 2.$$

The minimum value $C(n)$ We seek is then as stated at the beginning, since for

$$x_i = y_i^2 = c\left(\frac{w_i}{\sqrt{n-1}}\right)^2 = \frac{C\,.\,1}{n-1}$$

the original inequality reduces to equality.

Remark : The above shows $C(2) = 1$,

$$C(3) = \frac{2}{\sqrt{3}} \approx 1.1547,\ C(4) = \frac{3}{\sqrt[3]{4^2}} \approx 1.1905,$$

$C(5) = \dfrac{4}{\sqrt[4]{5^3}} \approx 1.1963$, and generally $C\ (n)\ \approx\ n^{1/(n-1)}$ which approaches 1 in the limit.

3. We consider polynomials $p(x)$ with coefficients in a field F of arbitrary characteristic and find as follows:

(i) If char $(F) = 0$, (in particular, if $F = R$), then $p(x) = a(x-3)^2$, where a is any scalar (possibly 0) in F;

(ii) If char $(F) = 2$, then every $p(x)$ satisfies the equation (clear);

(iii) If char $(F) = 2$ an odd prime, l, then there are infinitely many solutions, including all $p(x) = a(x-3)^2 (x^{l^v} - x + c)$ with $a, c \in F$, and $v = 0, 1, 2, \ldots$ (Note that $p(x)$ has the form $a(x-3)^2$ if $v = 0$.

To prove this, observe that if char $(F) \neq 2$, then $x - 1$ and $x - 3$ are coprime, whence $p(x) = (x-3)^2\, q(x)$ in $F[x]$.

Thus our equation becomes

$$(x^{l^v} - 1)^2 (x-3)^2 q(x) = (x-3)^2 (x-1)^2 q(x+2) \quad (*)$$

whence $q(x) = q(x+2)$, as polynomials; that is, elements of $F[x]$.

Now if char $(F) = 0$, then $(*)$ has only constant solution. (The most elementary proof of this: without loss of generally, $q(x) = x^n + ax^{n-1} + \ldots$. Then $q(x+2) - q(x) = 2nx^{n-1} + \ldots$, and this is non-zero if $n \geq 1$. Another proof: $(*)$ implies that $q(x)$ is periodic, which forces equations $q\ (x) = c$ to have infinitely many roots x, a contradiction).

This establishes the assertion (i).

Re: assertion (iii). Let char $(F) = l$ and

$q(x) = x^{l^v} - x + c$.

Then for $x = 0, 1, \ldots, l-1$, (that is for each element of the prime field), we have $q(x) = c$ and so $q(x) = q(x+1) = q(x+2) = \ldots$, yielding polynomials of degree greater than or equal to l which satisfy. $(*)$. This establishes the assertion (iii).

4. In this problem, we will prove that for $x, y, z \leq 0$,

$$0 \leq (yz + zx + xy)(x + y + z) - 2xyz \leq \frac{7}{27}(x + y + z)^3.$$

This is the homogeneous version of the original inequality. The expression in the middle expands to $\sum x^2 y + xyz$, which is clearly non-negative. We focus on the right inequality, which becomes $\sum x^2 y + xyz \leq \frac{7}{27} \sum x^3 + \frac{7}{9} \sum x^2 y + \frac{14}{9} \sum xyz$, which implies $6 \sum x^2 y \leq 7 \sum x^3 + 15xyz$.

A property of homogeneous polynomials, and an alternate definition, is the following: $p(x_1, x_2, \ldots, x_n)$ is homogeneous of degree k if

$p(\lambda x_1, \lambda x_2, \ldots, x_n) = \lambda^k p(x^1, x^2, \ldots, x_n)$ for all $\lambda \in R$.

Going back to the original problem,

$p(x, y, z) = (yz + zx + xy)(x + y + z) - 2xyz$.

If $x + y + z = 0$, then all three variables must be 0, and the inequality follows. Otherwise, we can set $\lambda = \dfrac{1}{x+y+z}$, and then

$$0 \leq p\left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right) \leq \frac{7}{27}$$

5. More generally, for any positive integer n, we claim that $\sqrt{4 + n\sqrt{4 + (n+2)\sqrt{4 + (n+4)\sqrt{\ldots}}}} = n + 2$, where the left side is defined as the limit of

$$F(n, m) = \sqrt{4 + n\sqrt{4 + (n+2)\sqrt{4 + (n+4)\sqrt{\ldots\sqrt{4 + m\sqrt{4}}}}}}$$

as $m \to \infty$ (where m is an integer and $(m - n)$ is even).
If $g(n, m) = F(n, m) - (n+2)$, we have
$F(n, m)^2 - (n + 2)^2 = (4 + nF(n + 2, m)) - (4 + n(n + 4))$
$\quad = n(F(n + 2, m) - (n + 4))$,

$$g(n,m) = \frac{n}{F(n,m)+n+2}\, g(n+2,m).$$

Clearly $F(n, m) > 2$,

So, $|g(n, m)| < \dfrac{n}{n+4}|g(n+2,m)|$.

By iterating this, we obtain

$$|g(n, m)| < \frac{n(n+2)}{m(m+2)}|g(m,m)| < \frac{n(n+2)}{m}.$$

Therefore $g(n,m) \to 0$ as $m \to \infty$

Let $S_n = \sqrt{4+(2n-1)\sqrt{4+(2n+1)\sqrt{4+(2n+3)\sqrt{\ldots}}}}$

S_n satisfies the recurrence relation

$S_n = \sqrt{4+(2n-1)S_{n+1}}$ if and only if

$(S_n - 2)(S_n + 2) = (S_n - 1)S_n + 1$.

By inspection, this admits $S_n = 2n + 1$ as a solution. We only have to prove that $S_1 = 3$ to make this induction complete. Let

$$T_n = \sqrt{4+\sqrt{4+3\sqrt{\ldots(2n-3)\sqrt{4+2n-1\sqrt{(2n+3)}}}}}$$

and $U_n = \sqrt{4+\sqrt{4+3\sqrt{\ldots(2n-3)\sqrt{4+(2n-1)(2n+3)}}}} = 3$

Clearly $T_n \le U_n$ and the latter is identically equal to 3. Therefore, using the fact that $B \ge A > 0$ implies that $\sqrt{(4+A)/(4+B)} \ge \sqrt{A/B}$,

$$1 \le \frac{T_n}{3} = \frac{T_n}{U_n} = \frac{\sqrt{4+\sqrt{\ldots+(2n-1)\sqrt{2n+3}}}}{\sqrt{4+\sqrt{\ldots(2n-1)\sqrt{2n+3}}}}$$

$$\ge \frac{\sqrt{\sqrt{\ldots+(2n-1)\sqrt{2n+3}}}}{\sqrt{\sqrt{\ldots+(2n-1)\sqrt{2n+3}}}} \ge \ldots \ge \sqrt[2^{n+1}]{\frac{1}{2n+3}}$$

$$\frac{1}{(2n+3)^{1/2^{n+1}}} \to 1$$

as $n \to \infty$ [for example, by rewriting as exp {– In $(2n + 3)/2^{n+1}$} and using L' Hopital's rule]. This proves that $S_1 = \lim_{n\to\infty} T_n = 3$. The required expression is precisely S_{14} and hence its value is 29.

6. Equal chords subtend equal angles at the centre of a circle; if each of sides of length i subtends an angle α_i (i = 1, 2, 3) at the centre of the given circle, then $2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 360°$,

where, $\dfrac{\alpha_1}{2} + \dfrac{\alpha_2}{2} = 90° - \dfrac{\alpha_3}{2}$,

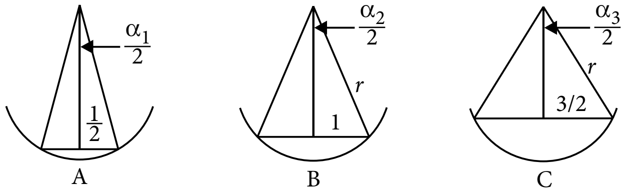
and $\cos\left(\dfrac{\alpha_1}{2} + \dfrac{\alpha_2}{2}\right) = \cos\left(90° - \dfrac{\alpha_3}{2}\right) = \sin\dfrac{\alpha_3}{2}$,

Next we apply the addition formula for the cosine:

$$\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2} = \sin\frac{\alpha_3}{2}, \qquad \ldots(i)$$

where, $\sin\dfrac{\alpha_1}{2} = \dfrac{1/2}{r}$ $\quad \cos\dfrac{\alpha_1}{2} = \dfrac{\sqrt{4r^2-1}}{2r}$;

$\sin\dfrac{\alpha_2}{2} = \dfrac{1}{r}$, $\quad \cos\dfrac{\alpha_2}{2} = \dfrac{\sqrt{r^2-1}}{r}$; $\sin\dfrac{\alpha_3}{2} = \dfrac{3/2}{r}$.



We substitute these expressions into (i) and obtain, after multiplying both sides by $2r^2$,

$$\sqrt{4r^2-1}.\sqrt{r^2-1} - 1 = 3r.$$

Now write it in the form $\sqrt{(4r^2-1)(r^2-1)} = 3r+1$, and sqaure, obtaining $(4r^2 - 1)(r^2 - 1) = 9r^2 + 6r + 1$, which is equivalent to $r(2r^3 - 7r - 3) = 0$.
Since $r \ne 0$, we have $2r^3 - 7r - 3 = 0$, which was to be shown.

7. (a) We immediately get $x_2 = 2$ and $x_3 = 2^k + 1$. Now we use mathematical induction for the proof. Assume that $x_0, x_1, \ldots, x_n$ are all natural numbers. We must show that $x_{n+1} \in N$. First we note that since $x_{n-2} \cdot x_n = x_{n-1}^k + 1$ it follows that x_{n-2} and x_{n-1} are relatively prime. Using $x_n = (x_{n-1}^k + 1)/x_{n-2}$ we infer that

$$x_{n+1} = \frac{x_n^k + 1}{x_{n-1}} = \frac{(x_{n-1}^k+1)^k + x_{n-2}^k}{x_{n-2}^k\, x_{n-1}}.$$

Thus obviously x_{n-2}^k divides $N = (x_{n-1}^k + 1)^k + x_{n-2}^k$ since x_n is a natural number. Furthermore, modulo x_{n-1} we have:
$N \equiv 1 + x^k_{n-2} = x_{n-3} \cdot x_{n-1} \equiv 0$.
That is, x_{n-1} also divides N and we are done.

(b) Now, $x_{n+1} = \dfrac{x_n^2+1}{x_{n-1}} \Leftrightarrow x_{n-1}\cdot x_{n+1} - x_n^2 = 1$.

That is, the sequence $\{y_n\} = \{x_{n-1} \cdot x_{n+1} - x_n^2\}$ is constant.
Setting $y_{n+1} = y_n$ we have
$x_n \cdot x_{n+2} - x_{n+1}^2 = x_{n-1} \cdot x_{n+1} - x_n^2$
$\Leftrightarrow x_n(x_n + x_{n+2}) = x_{n+1}(x_{n-1} + x_{n+1})$

$$\Leftrightarrow \frac{x_n + x_{n+2}}{x_{n+1}} = \frac{x_{n-1} + x_{n+1}}{x_n}.$$

That is, the sequence $\{z_n\} = \{(x_{n-1} + x_{n+1})/x_n\}$ is constant. From $z_1 = 3$ we get $(x_{n-1} + x_{n+1})/x_n = 3$; that is, $x_{n+1} = 3x_n - x_{n-1}$ for all $n \geq 1$, as claimed.

8. In general, $\sqrt{n+1} < f(n) < \frac{4}{3}\sqrt{n}$ ...(i)

for all even $n \geq 6$. In particular, for $n = 1992$, we would get $\sqrt{1993} < f(1992) < \frac{4}{3}\sqrt{1992}$.

First note that $f(n) = \frac{n}{f(n-1)} = \frac{n}{n-1} f(n-2)$ for all $n \geq 3$. If $N = 2k$ where $k \geq 2$, then multiplying $f(2q) = \frac{2q}{2q-1} f(2q-2)$ for q = 2, 3, ..., k, we get

$$f(2k) = \frac{4}{3}.\frac{6}{5}....\frac{2k}{2k-1}.f(2)$$

$$= \left(\frac{2}{1}\right).\left(\frac{4}{3}\right).\left(\frac{6}{5}\right)...\left(\frac{2k}{2k-1}\right) > \left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)...\left(\frac{2k+1}{2k}\right).$$

Hence, $(f(2k))^2 > \frac{2.4.6...2k}{1.3.5...(2k-1)}.\frac{3.5.7...(2k+1)}{2.4.6...2k} = 2k+1,$

$\therefore \quad f(n) = f(2k) > \sqrt{2k+1} = \sqrt{n+1}.$ ...(ii)

On the other hand, for $k \geq 3$ we have

$$2(2k) = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)...\left(\frac{2k-2}{2k-1}\right).2k$$

$$< \left(\frac{2}{3}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)...\left(\frac{2k-2}{2k}\right)2k.$$

Hence, $(f(2k))^2 < \left(\frac{2}{3}\right)^2.\frac{4.6...(2k-2)}{5.7...(2k-1)}.\frac{5.7...(2k-1)}{6.8...2k}.(2k)^2$

$$= \left(\frac{2}{3}\right)^2.4.2k$$

from which it follows that

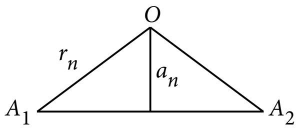
$\therefore \quad f(n) = f(2k) < \frac{4}{3}\sqrt{2k} = \frac{4}{3}\sqrt{n}.$ ...(iii)

The result follows from (ii) and (iii).

Note: Using similar arguments, upper and lower bounds for $f(n)$ when n is odd can also be easily derived. In fact, if we set $P = \frac{1.3.5...(2k-1)}{2.4.6..2k}$ (usually denoted by $\frac{2k-1!}{(2k)!}$, then various upper and lower bounds for P abound in the literature; for example, it is known that $\frac{1}{2}\sqrt{\frac{5}{4k+1}} \leq P \leq \frac{1}{2}\sqrt{\frac{3}{2k+1}}$

and $\frac{1}{\sqrt{\left(n+\frac{1}{2}\right)\pi}} < P \leq \frac{1}{\sqrt{n\pi}}.$

9. Let O be the centre of the regular n-gon. Let A_1A_2 denote one side of the regular n-gon



Then, we have $\angle A_1OA_2 = \frac{2\pi}{n}$, $\angle OA_1A_2 = \angle OA_2A_1 = \frac{\pi}{2} - \frac{\pi}{n}$. Thus, $|\overline{A_1A_2}| = \sqrt{r_n^2 + r_n^2 - 2r_n^2\cos\frac{2\pi}{n}}$

$$= \sqrt{2r_n^2(1-\cos\frac{2\pi}{n})} = \sqrt{4r_n^2\sin^2\frac{\pi}{n}} = 2r_n\sin\frac{\pi}{n}.$$

The circumference of the regular n-gon is

$2nr_n\sin\frac{\pi}{n} = 4$ whence $r_n = \frac{2}{n\sin\frac{\pi}{n}}$,

$$a_n = r_n\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = r_n\cos\frac{\pi}{n} = \frac{2}{n}\cot\frac{\pi}{n}.$$

In particular

$$r_4 = \frac{1}{2}\frac{1}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}}{2}, \quad a_4 = \frac{2}{4}\cot\frac{\pi}{4} = \frac{1}{2},$$

$$r_8 = \frac{2}{8\sin\frac{\pi}{8}} = \frac{1}{4\sin\frac{\pi}{8}}.$$

Now, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 1 - 2\sin^2\frac{\pi}{8}$

$\therefore \quad \sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}},$

So, $r_8 = \frac{1}{4}\frac{2}{\sqrt{2-\sqrt{2}}} = \frac{1}{2}.\frac{1}{\sqrt{2-\sqrt{2}}},$

and $a_8 = r_8\cos\frac{\pi}{8} = \frac{1}{4}\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = \frac{1}{4}\frac{1}{2-\sqrt{2}}\sqrt{2},$

since $\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1.$

For (b), $r_2 = 1$, $a_2 = 0$ as the 2-gon is a straight line with O lying at the middle of A_1 and A_2.

For (c), we have

$$a_n + r_n = r_n\left(1+\cos\frac{\pi}{n}\right) = 2r_n\cos^2\frac{\pi}{2n}$$

$$= \frac{4}{n\sin\frac{\pi}{n}}\cos^2\frac{\pi}{2n} = \frac{4}{2n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}\cos^2\frac{\pi}{2n} = \frac{2}{n}\cot\frac{\pi}{2n}.$$

Thus $\frac{1}{2}(a_n + r_n) = \frac{1}{n}\cot\left(\frac{\pi}{2n}\right) = a_{2n}$, and

$$a_{2n}r_n = \frac{1}{n}\frac{\cos\frac{\pi}{2n}}{n\sin\frac{\pi}{2n}}\cdot\frac{2}{n\sin\frac{\pi}{n}} = \frac{1}{n^2}\frac{\cos\frac{\pi}{2n}}{\sin^2\frac{\pi}{2n}\cos\frac{\pi}{2n}} = \frac{1}{n^2\sin^2\frac{\pi}{2n}},$$

so $\sqrt{a_{2n}r_n} = \dfrac{1}{n\sin\frac{\pi}{2n}} = r_{2n}$.

For (d), note $u_0 = 0$, $u_1 = 1$, and $u_2 = \frac{1}{2}$. For $n \geq 2$ we have that u_n is either the arithmetic or geometric mean of u_{n-1} and u_{n-2} and in either case lies between them. It is also easy to show by induction that u_0, u_2, u_4, ... form an increasing sequence, and u_1, u_3, u_5, ... form a decreasing sequence with $u_{2l} \leq u_{2s+1}$ for all l, $s \geq 0$. Let $\lim\limits_{k\to\infty} u_{2k} = P$ and $\lim\limits_{k\to\infty} u_{2k+1} = I$. Then $P \leq I$. We also have from $u_{2n} = \frac{1}{2}(u_{2n-1} + u_{2n-2})$ that $P = \frac{1}{2}(I + P)$ so that $I = P$ and $\lim\limits_{n\to\infty} u_n$ exists. Let $\lim\limits_{n\to\infty} u_n = L$. With $a_2 = 0$ and $r_2 = 1$, let $\bar{u}_{2k} = a_{2^{k+1}}$ and $\bar{u}_{2k+1} = r_{2^{k+1}}$, for $k = 0, 1, 2, ...$ From (c), $\bar{u}_0 = a_{2^1} = a_2 = 0$ and $\bar{u}_1 = r_{2^1} = r_2 = 1$. Also for $n = 2k + 2$, $\bar{u}_{2k+2}$

$$= a_{2^{k+1+1}} = a_{2.2^{k+1}} = \frac{1}{2}\left(a_{2^{k+1}} + b_{2^{k+1}}\right) = \frac{1}{2}(\bar{u}_{2k} + \bar{u}_{2k+1});$$

that is $\bar{u}_n = \frac{1}{2}(\bar{u}_{n-2} + \bar{u}_{n-1})$ and for $n = 2k + 3$

$$\bar{u}_{2k+3} = \bar{u}_{2(k+1)+1} = r_{2^{k+1+1}} = r_{2\left(2^{k+1}\right)}$$

$$= \sqrt{a_{2(2^{k+1})}.r_{2^{k+1}}} = \sqrt{a_{2^{k+1+1}}.r_{2^{k+1}}} = \sqrt{\bar{u}_{2(k+1)}\cdot\bar{u}_{2k+1}}$$

so $\bar{u}_n = \sqrt{\bar{u}_{n-1}\cdot\bar{u}_{n-2}}$. Thus u_n and $\bar{u}_n$ satisfy the same recurrence and it follows that $L = \lim\limits_{k\to\infty} a_{2^{k+1}} = \lim\limits_{k\to\infty} r_{2^{k+1}}$. Now, from the solution to (c),

$$r_n = \frac{2}{n\sin\frac{\pi}{n}} = \frac{2}{\pi}\frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$

So, $\lim\limits_{n\to\infty} r_n = \dfrac{2}{\pi}$ since $\dfrac{\pi}{n} \to 0$. Therefore $\lim\limits_{n\to\infty} u_n = \dfrac{2}{\pi}$.

10. From $a \geq b \geq c > 0$, we have

$$\frac{a+b}{c} \geq 2.\quad 0 < \frac{b+c}{a} \leq 2 \quad\text{and}\quad \frac{a+c}{b} \geq 1.$$

Now, $\dfrac{a^2 - b^2}{c} \geq 2(a - b)$, because $a \geq b$;

$\dfrac{c^2 - b^2}{a} \geq 2(c - b)$, because $c \leq b$

and $\dfrac{a^2 - c^2}{b} \geq a - c$, because $a \geq c$

After addition of these inequalities, we have

$$\frac{a^2 - b^2}{c} + \frac{c^2 - b^2}{a} + \frac{a^2 - c^2}{b} \geq 2(a - b) + 2(c - b) + (a - c),$$

that is, $\dfrac{a^2 - b^2}{c} + \dfrac{c^2 - b^2}{a} + \dfrac{a^2 - c^2}{b} \geq 3a - 4b + c.$

The equality holds if and only if $a = b = c > 0$.

CLASS XI | Series 4

CBSE DRILL

Synopsis and Chapter wise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2018-19.

Complex Numbers and Quadratic Equations | Linear Inequalities

Complex Numbers and Quadratic Equations

COMPLEX NUMBER

A number of the form $a + ib$ where a and b are real numbers and $i = \sqrt{-1}$ is called a complex number. It is usually denoted by z *i.e.*, $z = a + ib$.

Here, a is called real part and b is called imaginary part of z denoted by Re(z) and Im(z) respectively.

Note : Two complex numbers, $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are said to be equal if and only if $a_1 = a_2$ and $b_1 = b_2$ $\Leftrightarrow \mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

ALGEBRA OF COMPLEX NUMBERS

- **Addition of two complex numbers :** Let $z_1 = x + iy$ and $z_2 = u + iv$ be two complex numbers. Then the sum of z_1 and z_2 is given by $z_1 + z_2 = (x + u) + i(y + v)$
- **Difference of two complex numbers :** For two complex numbers z_1 and z_2, the difference of z_1 and z_2 is given by $z_1 - z_2 = z_1 + (-z_2)$.

Properties	
Closure law	$z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.
Commutative law	$z_1 + z_2 = z_2 + z_1$, where z_1 and z_2 are complex numbers.
Associative law	$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, where z_1, z_2 and z_3 are complex numbers.
Existence of additive identity	There exists a complex number $0 + i0$ (denoted as 0) which is called additive identity for every complex number.
Existence of additive inverse	For every complex number $z = x + iy$, we have $-x + i(-y)$ (denoted as $-z$). It is called additive inverse or negative of z.

- **Multiplication of two complex numbers :** Let $z_1 = x + iy$ and $z_2 = u + iv$ be any two complex numbers. Then the product z_1z_2 is defined as $z_1z_2 = (xu - yv) + i(xv + yu)$.

- **Division of two complex numbers :** For any two complex numbers z_1 and z_2 $(z_2 \neq 0)$, the quotient $\frac{z_1}{z_2}$ is defined as product of z_1 and $\frac{1}{z_2}$ *i.e.* $\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$.

Properties	
Closure law	The product of two complex numbers *i.e.*, $z_1\, z_2$ is also a complex number, where z_1 and z_2 are complex numbers.
Commutative law	For any two complex numbers z_1 and z_2, we have $z_1z_2 = z_2z_1$
Associative law	For any three complex numbers z_1, z_2, z_3, we have $(z_1z_2)z_3 = z_1(z_2z_3)$
Existence of multiplicative identity	There exists a complex number $1 + i0$ (denoted as 1), called the multiplicative identity such that $z \cdot 1 = z$, for every complex number z.
Existence of multiplicative inverse	For every non-zero complex number $z = x + iy$ $(x \neq 0, y \neq 0)$, we have the complex number $\frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the multiplicative inverse of z such that $z \cdot \frac{1}{z} = 1$ (the multiplicative identity)
Distributive law	For any three complex numbers z_1, z_2, z_3, we have (a) $z_1\,(z_2 + z_3) = z_1z_2 + z_1z_3$ (b) $(z_1 + z_2)\, z_3 = z_1z_3 + z_2z_3$

Note : $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. So, we can write as $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$ and $i^{4k+3} = -i$ for any integer k.

IDENTITIES OF COMPLEX NUMBERS

If z_1 and z_2 are two complex numbers, then

- $(z_1+z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$
- $(z_1-z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$
- $(z_1+z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$
- $(z_1-z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$
- $(z_1^2 - z_2^2) = (z_1+z_2)(z_1-z_2)$

MODULUS AND CONJUGATE OF COMPLEX NUMBERS

For any complex number, $z = a + ib$,

- Modulus of z denoted by $|z|$ is defined by $|z| = \sqrt{a^2+b^2}$
- Conjugate of z denoted by $\bar{z}$ is defined by $\bar{z} = a - ib$.

Properties of Modulus and Conjugate of Complex Numbers

(i) $\overline{(\bar{z})} = z$, $\text{Re}(z) = \text{Re}(\bar{z}) = \frac{z+\bar{z}}{2}$ and $\text{Im}(z) = \frac{z-\bar{z}}{2i}$

(ii) $|z_1z_2| = |z_1|\,|z_2|$ (iii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $|z_2| \neq 0$

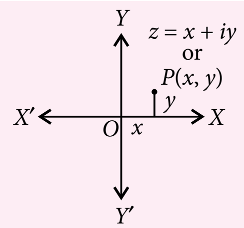
(iv) $|\overline{z_1z_2}| = |\bar{z}_1||\bar{z}_2|$ (v) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $(\bar{z}_2 \neq 0)$

(vii) $z^{-1} = \frac{\bar{z}}{|z|^2}$ (where z^{-1} is multiplicative inverse of z)

ARGAND PLANE

Any complex number $z = x + iy$ may be represented by a unique point P whose coordinates are (x, y). The plane on which complex numbers are represented is known as the Complex plane or Argand's plane.

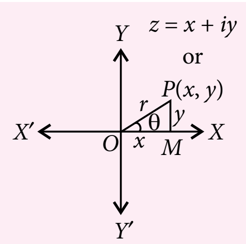


MPP-4 CLASS XI ANSWER KEY

1.	(a)	2.	(b)	3.	(d)	4.	(b)	5.	(d)
6.	(a)	7.	(a,b,c)	8.	(a,b,c)	9.	(a, c)	10.	(a)
11.	(a)	12.	(a,b,c,d)			13.	(a, c)	14.	(c)
15.	(b)	16.	(d)	17.	(7)	18.	(9)	19.	(2)
20.	(7)								

POLAR REPRESENTATION AND ARGUMENT OF COMPLEX NUMBERS

Let $z = x + iy$ be a complex number represented by $P(x, y)$ as shown.
Draw $PM \perp OX$ as shown in given figure. Then,
$OM = x$ and $PM = y$.
Join OP. Let $OP = r$ and
$\angle XOP = \theta$. Then,
$x = r\cos\theta$ and $y = r\sin\theta$.
$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$



$\Rightarrow r = \sqrt{x^2 + y^2} = |z|$

Also, $\tan\theta = \dfrac{y}{x}$

This form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is called polar form of the complex number z.
Also, angle θ is known as amplitude or agrument of z, written as $\arg(z)$.

Note : (i) The value of θ such that $-\pi < \theta \le \pi$ is called the principal argument of z.
(ii) The general value of the argument is $(2n\pi + \theta)$, where n is an integer and θ is the principal value of $\arg(z)$.

Properties of argument of complex numbers

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$, $(k = 0$ or 1 or $-1)$

(ii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$, $(k = 0$ or 1 or $-1)$

(iii) $\arg(zn) = n\arg(z) + 2k\pi$, $(k = 0$ or 1 or $-1)$

(iv) $\arg(\bar{z}) = -\arg(z)$

SQUARE ROOTS OF A COMPLEX NUMBER

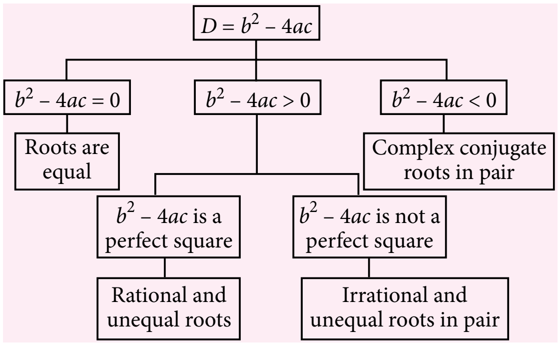
Let $z = a + ib$ be a complex number. Then

$$\sqrt{a+ib} = \begin{cases} \pm\left[\sqrt{\dfrac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\dfrac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right], & \text{when } b > 0 \\ \pm\left[\sqrt{\dfrac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} - i\sqrt{\dfrac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right], & \text{when } b < 0 \end{cases}$$

QUADRATIC EQUATIONS

An equation of the form $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \ne 0$ is known as quadratic equation. Then the solutions of the given equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$



Linear Inequalities

DEFINITION

A statement involving the symbols '>', '<', '≥' or '≤' is called an inequality.

TYPES OF INEQUALITIES

- Inequalities which do not involve variables are called numerical inequalities *e.g.*, $3 < 7$, $6 > 5$
- Inequalities which involve variables are called literal inequalities *e.g.*, $x \ge 4$, $y > 6$, $x - y < 0$
- An equation of the form $ax + b < 0$, or $ax + b \ge 0$, or $ax + by > 0$ etc. are known as linear inequations *e.g.*, $x + 3 < 0$, $3x + 2y > 7$
- An equation of the form $ax^2 + bx + c < 0$ or $ax^2 + bx + c \le 0$ or $ax^2 + bx + c > 0$ or $ax^2 + bx + c \ge 0$ is known as a quadratic inequation *e.g.*, $x^2 + 7x + 4 > 0$
- Inequalities involving the symbols '>' or '<' are called strict inequalities *e.g.*, $x + y > 5$, $y < 0$
- Inequalities involving the symbols '≥' or '≤' are called slack inequalities *e.g.*, $4x + 3y \ge 2$, $x \le -4$

ALGEBRAIC SOLUTIONS OF LINEAR INEQUALITIES IN ONE VARIABLE AND THEIR GRAPHICAL REPRESENTATION

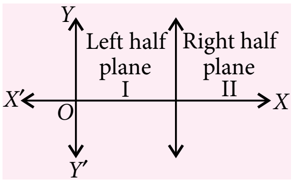
The solution of an inequality in one variable is the value of the variable which makes it a true statement.

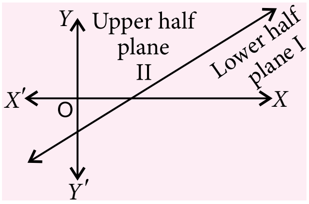
Rules for solving inequalities

(i) Same number or expression may be added to or subtracted from both sides of an inequality without affecting the sign of inequality.
(ii) Both sides of an inequality can be multiplied or divided by the same positive number without affecting the inequality sign.
(iii) When both sides of an inequation is multiplied or divided by a negative number, then sign of inequality is reversed.

(iv) Inequalities of the type $x < a$ (or $x > a$) on a number line is represented graphically by putting a circle on the number a and drawing dark line to the left or right of 'a'. Similarly for $x \le a$ (or $x \ge a$) we put dark circle (showing 'a' also satisfies the inequality) and then draw dark line to the left or right of a.

GRAPHICAL SOLUTION OF LINEAR INEQUALITIES IN TWO VARIABLES





The region containing all the solutions of an inequality is called the solution region.

- In order to identify the half plane represented by an inequality, it is just sufficient to take any point (a, b) not on the line and check whether it satisfies the inequality or not. If it satisfies, then the inequality represents the half plane which contains the point, otherwise the inequality represents the half plane which does not contain the point.
- In case of inequalities of type $ax + by \le$ (or $\ge$) c, the points on the line $ax + by = c$ are also included in the solution region. But if inequality contains only sign '<' or '>' then points on the line (equation) are not included in solution region.

Note : In case of system of linear inequalities in two variables, we represent each inequation graphically and then common region represented by all the inequations is the solution region.

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Show that for every complex number $z = x + iy \ne 0 + i\cdot 0$; there exists a complex number $z_1 = \dfrac{x-iy}{x^2+y^2}$ such that $zz_1 = 1$
2. Solve : $|x + 1| > 4, x \in R$.
3. Write the complex number $z = \dfrac{2+i}{(1+i)(1-2i)}$ in standard form.
4. Solve : $5x < 24, x \in I$ and represent it graphically.
5. Solve the following equation by factorization method : $x^2 - ix + 6 = 0$.

SHORT ANSWER TYPE

6. Show that the points representing the complex numbers $(3 + 2i)$, $(2 - i)$ and $-7i$ are collinear.
7. Express $(4 - 3i)^3$ in the standard form.
8. Find roots of $x^2 - 14x + 58 = 0$.
9. Solve the equation $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}\,i = 0$ using factorization method.
10. Solve the inequation : $-3 \le 3 - 2x < 9, x \in R$.

LONG ANSWER TYPE - I

11. Solve : $2x - 1 > x + \dfrac{7-x}{3} > 2$, $x \in R$ and represent it graphically.
12. Represent the complex number $(-1 + i\sqrt{3})$ in the polar form.
13. Solve : $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
14. Show that $\dfrac{(1+i)(3+i)}{(3-i)} - \dfrac{(1-i)(3-i)}{(3+i)} = \dfrac{14}{5}i$
15. Evaluate : $\sqrt{16 - 30i}$.

LONG ANSWER TYPE - II

16. Solve the inequation : $\dfrac{|x+1| - x}{x} < 1$.
17. Reduce $\left(\dfrac{1}{1-2i} + \dfrac{3}{1+i}\right)\left(\dfrac{3+4i}{2-4i}\right)$ to the standard form.
18. For complex values of z, solve $|z| + z = (2 + i)$.
19. Solve the following system of inequations graphically:
 $3x + 2y \le 24$, $x + 2y \le 16$, $x + y \le 10$, $x \ge 0$, $y \ge 0$
20. If $|z_1| = |z_2| = |z_3| = ... = |z_n| = 1$, then prove that
 $\left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} + ... + \dfrac{1}{z_n}\right| = |z_1 + z_2 + z_3 + ... + z_n|$

SOLUTIONS

1. $zz_1 = (x+iy)\left(\dfrac{x-iy}{x^2+y^2}\right)$

$= (x+iy)\left(\dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}\right)$

$= \left(\dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2}\right) + i\left(\dfrac{-xy}{x^2+y^2} + \dfrac{xy}{x^2+y^2}\right)$

$= 1 + i\,.\,(0) = 1.$

2. We know, $|x| > a \Leftrightarrow x < -a$ or $x > a$

$\therefore$ $|x + 1| > 4 \Rightarrow x + 1 < -4$ or $x + 1 > 4$

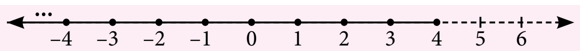
$\Rightarrow$ $x < -5$ or $x > 3$

$\therefore$ Solution set = $\{x \in R : x < -5\} \cup \{x \in R : x > 3\}$
= $(-\infty, -5) \cup (3, \infty)$.

3. We have, $z = \dfrac{2+i}{(1+i)(1-2i)} = \dfrac{2+i}{3-i} = \dfrac{(2+i)(3+i)}{(3-i)(3+i)}$

$= \dfrac{5+5i}{10} = \dfrac{1}{2} + i\dfrac{1}{2}$

4. Solution set = $\{x \in I : 5x < 24\} = \{x \in I : x < 4.8\}$
= $\{4, 3, 2, 1, 0, -1, -2, -3, ...\}$
On the number line, we may represent it as shown below.



The darkened circles show the integers contained in the set.

5. Given, $x^2 - ix + 6 = 0$
$\Rightarrow x^2 - 3ix + 2ix + 6 = 0 \quad \Rightarrow x(x - 3i) + 2i(x - 3i) = 0$
$\Rightarrow (x + 2i)(x - 3i) = 0 \quad \Rightarrow x = -2i$ or $x = 3i$
Hence, the roots of the given equation are $-2i$ and $3i$.

6. Let these numbers be represented on the argand plane by the points A, B and C respectively.
Then, $AB = |(2 - i) - (3 + 2i)| = |-1 - 3i| = \sqrt{10}$
$BC = |-7i - (2 - i)| = |-2 - 6i| = \sqrt{40} = 2\sqrt{10}$
and $AC = |-7i - (3+2i)| = |-3 - 9i| = \sqrt{90} = 3\sqrt{10}$
Clearly, $AB + BC = AC$
Hence, the points A, B, C are collinear.

7. We have, $(4 - 3i)^3 = 4^3 - (3i)^3 - 3 \times 4 \times 3i \times (4 - 3i)$
$= 64 - 27i^3 - 36i(4 - 3i) = (64 - 108) + i(27 - 144)$
$= (-44 - 117i)$
$\therefore \quad (4 - 3i)^3 = (-44 - 117i)$

8. We have, $x^2 - 14x + 58 = 0$
The two complex roots of the given equation are given by

$\alpha = \dfrac{14 + \sqrt{(14)^2 - 4 \cdot 1 \cdot 58}}{2}$ and $\beta = \dfrac{14 - \sqrt{(14)^2 - 4 \cdot 1 \cdot 58}}{2}$

$\Rightarrow \alpha = \dfrac{14 - i\sqrt{232 - 196}}{2}$ and $\beta = \dfrac{14 + i\sqrt{232 - 196}}{2}$

$\Rightarrow \alpha = \dfrac{14 - 6i}{2}$ and $\beta = \dfrac{14 + 6i}{2}$

$\Rightarrow \alpha = 7 - 3i$ and $\beta = 7 + 3i$
Hence, the roots of the given equation are $7 - 3i$ and $7 + 3i$.

9. We have, $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}\,i = 0$
$\Rightarrow (x^2 - 3\sqrt{2}x) - 2ix + 6\sqrt{2}\,i = 0$
$\Rightarrow x(x - 3\sqrt{2}) - 2i(x - 3\sqrt{2}) = 0$
$\Rightarrow (x - 2i)(x - 3\sqrt{2}) = 0$
$\Rightarrow x = 2i$ or $x = 3\sqrt{2}$
Hence, the roots of the given equation are $2i$ and $3\sqrt{2}$.

Only 14 girls in top 500 of JEE Advanced

A mere fourteen girls have made the cut to the top 500 ranks of the IIT JEE Advanced exam, underscoring the gender divide in technical education at the elite IITs. The number of females rises to just 46 even when the list is expanded to include the top 1,000 (as compared to 68 last year).

However, under the HRD ministry's gender diversity plan, at least 8% more seats (800 in all) will be added to IIT this year to accommodate more girls, thus enhancing female representation in popular streams like computer science and electrical engineering. For instance, the seven older IITs will have 3% girls in computer science with the female-only seats.

Data from IIT-Kanpur shows 3,000-odd girls have been shortlisted by the Joint Admission Board from the top 24,500 ranks. Among the top 5,000 students, there are 410 girls, and in the top 10,000 ranks of the common rank list, there are 935 of them. Excluding the girls-only quota, the 23 IITs have 11,279 seats; the number of girls in the top 12,000 are about 1,202.

Girls SCORE in JEE (Advanced)

TOP 1,000	46
Top 5,000	410
Top 10,000	935
Top 12,000	1,202

A JEE chairman pointed out that mandatory reservation and addition of seats for girls was to ensure "14% girls in every programme".

According to the IITs, female candidates are eligible for a seat from the female-only pool as well as the gender-neutral pool of a program. A female candidate can compete for a seat in the gender-neutral pool only if she fails to get a seat from the female-only pool.

"But if you see the number of female candidates in the top ranks, they are very few and most will opt for the female-only pool to get into a popular course and a better institute," said an IIT Bombay faculty member.

Under business rules set by the IIT for seat allocation, the 800-odd seats for females will also follow quota norms. For example, consider an OBC-NCL female candidate with a general rank. She will be first considered for a seat from the female-only pool of general seats, followed by the gender-neutral pool of general seats for that programme. If she does not make it, she will be eligible under the OBC category.

Several attempts have been made in the past to ensure a larger share of girls at the IITs. Even the admission form's cost was reduced on the C N R Rao committee's recommendations. But that did not boost the numbers. Next year, according to the decision of the Joint Admission Board, more seats would be added to ensure that girls constitute 17% of total students. By 2020, the ministry aims to increase percentage of girl students at IITs to 20.

10. We have, $-3 \leq 3 - 2x < 9 \Rightarrow -3 \leq 3 - 2x$ and $3 - 2x < 9$.
Now, $-3 \leq 3 - 2x$

$\Rightarrow \; -3 + 2x \leq 3 \;\Rightarrow\; 2x \leq 6 \;\Rightarrow\; x \leq 3$ ...(i)

Again, $3 - 2x < 9$

$\Rightarrow \; -2x < 6 \;\Rightarrow\; x > -3$ ...(ii)

From (ii) and (i), we get $-3 < x \leq 3$.

$\therefore$ Solution set $= \{x \in R : -3 < x \leq 3\}$

11. Given, $2x - 1 > x + \frac{7-x}{3} > 2$

$\Rightarrow \; 2x - 1 > x + \frac{7-x}{3}$ and $x + \frac{7-x}{3} > 2$

Now, $2x - 1 > x + \frac{7-x}{3} \Rightarrow 6x - 3 > 3x + 7 - x$

[Multiplying both sides by 3]

$\Rightarrow \; 6x - 3 > 2x + 7 \;\Rightarrow\; 4x - 3 > 7$

$\Rightarrow \; 4x > 10 \;\Rightarrow\; x > \frac{5}{2}$ ...(i)

Again, $x + \frac{7-x}{3} > 2 \Rightarrow 3x + 7 - x > 6$

[Multiplying both sides by 3]

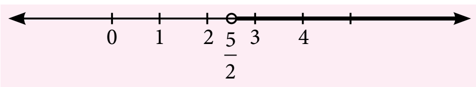
$\Rightarrow \; 2x + 7 > 6 \;\Rightarrow\; 2x > -1 \;\Rightarrow\; x > \frac{-1}{2}$ ...(ii)

From (ii) and (i), we get $x > -\frac{1}{2}$ and $x > \frac{5}{2}$ respectively.

So, we must have $x > \frac{5}{2}$.

$\therefore$ Solution set $= \{x \in R : x > \frac{5}{2}\}$

Graph of this set is given below.



12. Let $z = (-1 + i\sqrt{3})$. Now, $(-1, \sqrt{3})$ lies in quadrant II.
Let $z = r(\cos\theta + i\sin\theta)$.
Then, $r\cos\theta = -1$ and $r\sin\theta = \sqrt{3}$
On squaring and adding, we get $r^2 = 4$

$\therefore \; r = 2$

$\therefore \; \cos\theta = \frac{-1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2} \;\Rightarrow\; \tan\theta = -\sqrt{3}$

Now, $\tan\alpha = |\tan\theta| = \sqrt{3} \;\Rightarrow\; \alpha = \frac{\pi}{3}$

Now, $\alpha = \frac{\pi}{3}$ and z lies in quadrant II.

$\therefore \; \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{3}\right) = \frac{2\pi}{3}$

Thus, $z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ is the required polar form.

13. The given equation is of the form $ax^2 + bx + c = 0$
Here, $a = \sqrt{3}$, $b = -\sqrt{2}$ and $c = 3\sqrt{3}$

$\therefore$ The roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{2} \pm \sqrt{(-\sqrt{2})^2 - 4 \times \sqrt{3} \times 3\sqrt{3}}}{(2 \times \sqrt{3})}$$

$$= \frac{\sqrt{2} \pm \sqrt{2 - 36}}{2\sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{-34}}{2\sqrt{3}} = \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}$$

Hence, the roots of given equation are

$\frac{\sqrt{2} + i\sqrt{34}}{2\sqrt{3}}$ and $\frac{\sqrt{2} - i\sqrt{34}}{2\sqrt{3}}$.

14. We have, $\frac{(1+i)(3+i)}{(3-i)} = \frac{(3-1)+(3+1)i}{(3-i)}$

$$= \frac{(2+4i)}{(3-i)} = \frac{(2+4i)}{(3-i)} \times \frac{(3+i)}{(3+i)}$$

$$= \frac{(2+4i)(3+i)}{(9+1)} = \frac{(6-4)+(12+2)i}{10} = \frac{(2+14i)}{10} = \frac{(1+7i)}{5}$$

Also, $\frac{(1-i)(3-i)}{(3+i)} = \frac{(3-1)-4i}{(3+i)}$

$$= \frac{(2-4i)}{(3+i)} \times \frac{(3-i)}{(3-i)} = \frac{(6-4)-14i}{(9+1)} = \frac{(2-14i)}{10} = \frac{(1-7i)}{5}$$

$\therefore$ Given expression $= \frac{(1+7i)}{5} - \frac{(1-7i)}{5}$

$$= \frac{(1+7i)-(1-7i)}{5} = \frac{14}{5}i$$

15. Let $\sqrt{16 - 30i} = x - iy$ ...(i)

On squaring both sides of (i), we get

$(16 - 30i) = (x - iy)^2$

$\Rightarrow (16 - 30i) = (x^2 - y^2) - i(2xy)$ ...(ii)

On comparing real part and imaginary part, we get

$x^2 - y^2 = 16$ and $2xy = 30$

$\therefore \; (x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{(16)^2 + (30)^2}$

$= \sqrt{256 + 900} = \sqrt{1156} = 34$

Thus, $x^2 - y^2 = 16$ ...(iii)

and $x^2 + y^2 = 34$ ...(iv)

On solving (iii) and (iv), we get : $x^2 = 25$ and $y^2 = 9$

$\therefore \; x = \pm 5$ and $y = \pm 3$

Since $xy > 0$, so x and y are of the same sign.

$\therefore \; (x = 5, y = 3)$ or $(x = -5, y = -3)$

Hence, $\sqrt{16 - 30i} = (5 - 3i)$ or $(-5 + 3i)$

16. We have, $\frac{|x+1| - x}{x} < 1$

Now, $x + 1 = 0 \Rightarrow x = -1$

Case I. When $x \leq -1$:

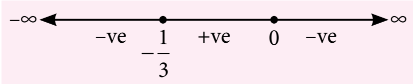
In this case, $x + 1 \le 0 \Rightarrow |x + 1| = -(x + 1)$

Now, $\dfrac{|x+1|-x}{x} < 1 \Rightarrow \dfrac{-(x+1)-x}{x} < 1$

$\Rightarrow \dfrac{-2x-1}{x} < 1 \quad \Rightarrow \dfrac{-2x-1}{x} - 1 < 0$

$\Rightarrow \dfrac{-2x-1-x}{x} < 0 \Rightarrow \dfrac{-3x-1}{x} < 0$.

Sign scheme for $\dfrac{-3x-1}{x}$ is



$\Rightarrow -\infty < x < -\dfrac{1}{3}$ or $0 < x < 1$

But in this case, $x \le -1 \quad \Rightarrow -\infty < x \le -1$ ...(i)

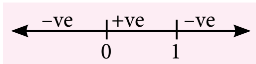
Case II. When $x > -1$:

In this case, $x + 1 > 0 \quad \Rightarrow |x + 1| = x + 1$

Now, $\dfrac{|x+1|-x}{x} < 1 \quad \Rightarrow \dfrac{x+1-x}{x} < 1$

$\Rightarrow \dfrac{1}{x} < 1 \quad \Rightarrow \dfrac{1}{x} - 1 < 0 \Rightarrow \dfrac{1-x}{x} < 0$

Sign scheme for $\dfrac{1-x}{x}$ is



$\Rightarrow \quad -\infty < x < 0$ or $1 < x < \infty$

But in this case, $x > -1$

$\Rightarrow \quad -1 < x < 0$ or $1 < x < \infty$ ...(ii)

From (i) and (ii), all possible values of x are given by

$-\infty < x \le -1$ or $-1 < x < 0$ or $1 < x < \infty$

$\therefore$ Solution set $= (-\infty, 0) \cup (1, \infty)$

17. We have, $\dfrac{1}{(1-2i)} = \dfrac{1}{(1-2i)} \times \dfrac{(1+2i)}{(1+2i)}$

$= \dfrac{(1+2i)}{(1^2-4i^2)} = \dfrac{(1+2i)}{5} = \left(\dfrac{1}{5} + \dfrac{2}{5}i\right)$

Also, $\dfrac{3}{(1+i)} = \dfrac{3}{(1+i)} \times \dfrac{(1-i)}{(1-i)}$

$= \dfrac{3(1-i)}{(1^2-i^2)} = \dfrac{3(1-i)}{2} = \left(\dfrac{3}{2} - \dfrac{3}{2}i\right)$

Now, $\dfrac{(3+4i)}{(2-4i)} = \dfrac{(3+4i)}{(2-4i)} \times \dfrac{(2+4i)}{(2+4i)}$

$= \dfrac{(6-16)+(12+8)i}{(4-16i^2)} = \dfrac{-10+20i}{20} = \left(\dfrac{-1}{2} + i\right)$

$\therefore$ Given expression $= \left\{\left(\dfrac{1}{5} + \dfrac{2}{5}i\right) + \left(\dfrac{3}{2} - \dfrac{3}{2}i\right)\right\}\left(-\dfrac{1}{2} + i\right)$

$$= \left\{\left(\frac{1}{5}+\frac{3}{2}\right)+\left(\frac{2}{5}-\frac{3}{2}\right)i\right\}\left\{\frac{-1+2i}{2}\right\}$$

$$= \left(\frac{17}{10}-\frac{11}{10}i\right)\left(\frac{-1+2i}{2}\right)=\frac{(17-11i)}{10}\times\frac{(-1+2i)}{2}$$

$$= \frac{(-17+22)+(34+11)i}{20}=\frac{(5+45i)}{20}$$

$$= \left(\frac{5}{20}+\frac{45}{20}i\right)=\left(\frac{1}{4}+\frac{9}{4}i\right)$$

18. Let $z = (x + iy)$. Then, $|z| + z = (2 + i)$

$\Rightarrow$ $|x + iy| + (x + iy) = (2 + i)$

$\Rightarrow$ $\left\{\sqrt{x^2+y^2}+x\right\}+iy=(2+i)$

$\Rightarrow$ $\sqrt{x^2+y^2}+x=2$ and $y=1$

[Equating real and imaginary parts separately]

$\Rightarrow$ $\sqrt{x^2+y^2}=(2-x)$ and $y=1$

$\Rightarrow$ $\sqrt{x^2+1}=(2-x)$ and $y=1$ $\qquad [\because y = 1]$

$\Rightarrow$ $(x^2 + 1) = (2 - x^2)$ and $y = 1$

$\Rightarrow$ $(x^2 + 1) = (4 + x^2 - 4x)$ and $y = 1$

$\Rightarrow$ $4 - 4x = 1$ and $y = 1$ $\Rightarrow$ $x=\frac{3}{4}$ and $y=1$.

Hence, $z=\left(\frac{3}{4}+i\right)$.

19. Given inequations are

$3x + 2y \le 24$ ...(i), $x + 2y \le 16$ ...(ii),

$x + y \le 10$ ...(iii), $x \ge 0$ ...(iv)

and $y \ge 0$ ...(v)

Equations corresponding to inequations (i), (ii) and (iii) are

$3x + 2y = 24$ ...(vi), $x + 2y = 16$ ...(vii)

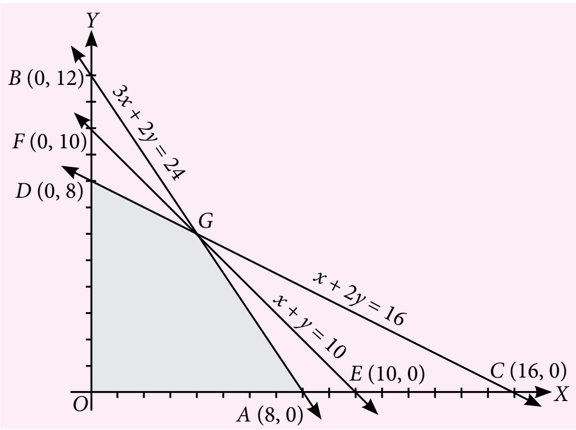
and $x + y = 10$ ...(viii)

Line (vi) cuts x-axis at $A(8, 0)$ and y-axis at $B(0, 12)$

Line (vii) cuts x-axis at $C(16, 0)$ and y-axis at $D(0, 8)$

Line (viii) cuts x-axis at $E(10, 0)$ and y-axis at $F(0, 10)$

Graph of given equation is represented as shown:



Inequations $x \ge 0$ and $y \ge 0$ represents the solution in 1st quadrant only.

$\therefore$ Solution set *i.e.*, set of all points in the common region is represented by $ODGA$.

It is the set of all points in the shaded region.

20. We have, $|z_1| = |z_2| = |z_3| = ... = |z_n| = 1$

$\Rightarrow$ $|z_1|^2 = |z_2|^2 = |z_3|^2 = ... = |z_n|^2 = 1$

$\Rightarrow$ $z_1\bar{z}_1=1, z_2\bar{z}_2=1, z_3\bar{z}_3=1, ..., z_n\bar{z}_n=1$

$\Rightarrow$ $\frac{1}{z_1}=\bar{z}_1, \frac{1}{z_2}=\bar{z}_2, \frac{1}{z_3}=\bar{z}_3, ..., \frac{1}{z_n}=\bar{z}_n$

$\Rightarrow$ $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+...+\frac{1}{z_n}\right|=|\bar{z}_1+\bar{z}_2+\bar{z}_3+...+\bar{z}_n|$

$= |\overline{z_1+z_2+z_3+...+z_n}|$

$= |z_1+z_2+z_3+...+z_n|$ $\qquad [\because |\bar{z}|=|z|]$

$\therefore$ $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+...+\frac{1}{z_n}\right|$

$=|z_1+z_2+z_3+...+z_n|$

◇◇

SAMURAI SUDOKU

SOLUTION - JULY 2018

5	1	8	7	9	4	6	3	2				6	2	9	4	5	3	7	8	1
9	6	7	2	3	5	4	1	8				1	5	7	9	8	6	3	2	4
2	4	3	6	1	8	7	5	9				4	8	3	7	1	2	6	9	5
7	3	5	4	2	6	9	8	1				9	4	5	3	2	7	1	6	8
4	8	6	1	5	9	2	7	3				3	1	8	5	6	4	9	7	2
1	9	2	3	8	7	5	6	4				7	6	2	8	9	1	4	5	3
8	2	4	5	7	1	3	9	6	1	5	2	8	7	4	6	3	5	2	1	9
3	7	9	8	6	2	1	4	5	3	7	8	2	9	6	1	4	8	5	3	7
6	5	1	9	4	3	8	2	7	9	6	4	5	3	1	2	7	9	8	4	6
						9	8	3	2	4	6	7	1	5						
						5	1	4	8	9	7	3	6	2						
						6	7	2	5	3	1	9	4	8						
7	1	8	2	3	6	4	5	9	6	2	3	1	8	7	3	2	9	6	4	5
2	4	6	8	9	5	7	3	1	4	8	5	6	2	9	7	4	5	8	1	3
9	3	5	4	1	7	2	6	8	7	1	9	4	5	3	1	6	8	2	7	9
3	8	2	9	5	4	1	7	6				2	6	5	4	3	1	7	9	8
4	6	9	1	7	3	8	2	5				7	1	4	8	9	6	5	3	2
1	5	7	6	2	8	9	4	3				9	3	8	5	7	2	1	6	4
8	9	4	5	6	2	3	1	7				8	4	6	2	1	3	9	5	7
6	2	3	7	8	1	5	9	4				3	9	2	6	5	7	4	8	1
5	7	1	3	4	9	6	8	2				5	7	1	9	8	4	3	2	6

MPP-4 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.

Self check table given at the end will help you to check your readiness.

Permutations and Combinations, Mathematical Reasoning, Mathematical Induction

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. A person always prefers to eat 'parantha' and 'vegetable dish' in his meal. How many ways can he make his platter in a marriage party, if there are three types of 'paranthas', four types of 'vegetable dish', three types of 'salads' and two types of 'sauces'?

(a) 3360 (b) 4096

(c) 3000 (d) none of these

2. For all positive integers $n > 1$,

$\{x(x^{n-1} - na^{n-1}) + a^n(n-1)\}$ is divisible by

(a) $(x-a)^2$ (b) $x - a$

(c) $2(x-a)$ (d) $x + a$

3. The number of ways in which 10 candidates $A_1, A_2, ..., A_{10}$ can be ranked such that A_1 is always above A_{10} is

(a) 5! (b) 2(5!)

(c) 0 (d) $\frac{1}{2}(10!)$

4. The proposition $(p \to \sim p) \wedge (\sim p \to p)$ is

(a) a tautology

(b) a contradiction

(c) neither a tautology nor a contradiction

(d) a tautology and a contradiction

5. The maximum number of points of intersection of 6 circles is

(a) 25 (b) 24 (c) 50 (d) 30

6. If p : a man is happy.

q : a man is rich.

Then, the statement, "If a man is not happy, then he is not rich" is written as

(a) $\sim p \to \sim q$ (b) $\sim q \to p$

(c) $\sim q \to \sim p$ (d) $q \to \sim p$

One or More Than One Option(s) Correct Type

7. If p, q and r are simple propositions, then $(\sim p \vee q) \Rightarrow r$ is true, when p, q and r are, respectively

(a) T, F, T (b) T, T, T

(c) F, T, T (d) F, F, F

8. A fair coin is tossed n times. Let a_n denote the number of cases in which no two heads occur consecutively. Then which of the following is true?

(a) $a_1 = 2$ (b) $a_2 = 3$

(c) $a_5 = 13$ (d) $a_6 = 55$

9. Two players P_1 and P_2 plays a series of $2n$ games. Each game can result in either a win or loss for P_1. Total number of ways in which P_1 can win the series of these games, is equal to

(a) $\frac{1}{2}(2^{2n} - {}^{2n}C_n)$ (b) $\frac{1}{2}(2^{2n} - 2 \cdot {}^{2n}C_n)$

(c) $\frac{1}{2}\left(2^n - \frac{2n!}{n!\,n!}\right)$ (d) none of these

10. For all positive integers n, $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is a/an

(a) integer (b) rational number

(c) irrational number (d) none of these

11. The proposition $\sim (p \Rightarrow q) \Rightarrow (\sim p \vee \sim q)$ is

(a) a tautology (b) a contradiction

(c) either (a) or (b) (d) neither (a) nor (b)

12. All the five-digit numbers in which each successive digit exceeds its predecessor are arranged in the increasing order. The 105th number does not contain the digit

(a) 1 (b) 3 (c) 4 (d) 5

13. If p and q are simple propositions, then $(\sim p \wedge q) \vee (\sim q \wedge p)$ is false when p and q are respectively

(a) T, T (b) F, T

(c) F, F (d) none of these

Comprehension Type

Two numbers x and y are drawn without replacement from the set of the first 15 natural numbers. The number of ways of drawing the numbers such that

14. $x^3 + y^3$ is divisible by 3, is

(a) 21 (b) 33 (c) 35 (d) 69

15. $x^2 - y^2$ is divisible by 5, is

(a) 21 (b) 33 (c) 35 (d) 69

Matrix Match Type

16. Match the following:

	Column-I		Column-II
P.	The greatest positive integer, which divides $(n + 2)(n + 3)(n + 4)(n + 5)(n + 6)$ for all $n \in N$, is	1.	120
Q.	A man moves one unit distance for each step he takes either along the positive x-direction or negative x-direction. The number of ways he moves from the point (0, 0) to (2, 0) in at most 10 steps is	2.	3720
R.	Given 5 different green dyes, 4 different blue dyes and 3 different red dyes, the number of combinations of dyes taking atleast one green dye and one blue dye, is	3.	286
S.	Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to	4.	7200

	P	Q	R	S
(a)	1	3	4	2
(b)	2	4	1	3
(c)	4	2	3	1
(d)	1	3	2	4

Integer Answer Type

17. For each natural number, the statement $P(n) = 2^{3n} - 1$ is divisible by

18. The number of ordered pairs of natural numbers (a, b) such that $\frac{ab}{a+b} = 511$ is

19. The number of ways in which we can choose 5 letters from the word 'INTERNATIONAL' is N. Then $\frac{N}{128}$ is equal to

20. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then value of n is

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT BOOSTERS

Matrices and Determinants

Class XII

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

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MATRICES

DEFINITION

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed within small ( ) or big [ ] brackets. The numbers are called the elements or entries of the matrix.

ORDER OF A MATRIX

A matrix having m rows and n columns is called a matrix of order $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1j} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2j} & \dots a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{i1} & a_{i2} & a_{i3} & \dots a_{ij} & \dots a_{in} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & a_{m3} & \dots a_{mj} & \dots a_{mn} \end{bmatrix}_{m\times n} \text{ or } A = [a_{ij}]_{m\times n},$$

where $i = 1, 2, \dots\dots, m$ and $j = 1, 2, \dots\dots, n$.

TYPES OF MATRICES

(1) **Row matrix :** A matrix is said to be a row matrix if it has only one row and any number of columns.

(2) **Column matrix :** A matrix is said to be a column matrix if it has only one column and any number of rows.

(3) **Singleton matrix :** If in a matrix there is only one element then it is called singleton matrix.
$A = [a_{ij}]_{m\times n}$ is a singleton matrix, if $m = n = 1$

(4) **Null or zero matrix :** If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by O.
$A = [a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij} = 0$ for all i and j.

(5) **Square matrix:** If number of rows and number of columns in a matrix are equal, then it is called a square matrix.
$A = [a_{ij}]_{m\times n}$ is a square matrix if $m = n$.

(6) **Rectangular matrix :** If in a matrix number of rows and number of columns are not equal, then it is called rectangular matrix.
$A = [a_{ij}]_{m\times n}$ is a rectangular matrix if $m \neq n$.

(7) **Diagonal matrix :** If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix. Thus a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

(8) **Equal matrices :** Two matrices A and B are said to be equal if and only if they are of same order and their corresponding elements are equal.

(9) **Identity matrix :** A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix. We denote the identity matrix of order n by I_n.

(10) **Scalar matrix :** A square matrix whose all non diagonal elements are zero and diagonal elements are equal is called a scalar matrix.

(11) **Triangular matrix :** A square matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types :

- **Upper triangular matrix :** A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij} = 0$ when $i > j$.
- **Lower triangular matrix :** A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij} = 0$ when $i < j$.

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He trains IIT and Olympiad aspirants.

TRACE OF A MATRIX

The sum of diagonal elements of a square matrix A is called the trace of matrix A, which is denoted by $tr(A)$.

$$tr\ (A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$$

Properties of trace of a matrix

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ and λ be a scalar

- $tr(\lambda A) = \lambda\ tr(A)$
- $tr(A - B) = tr(A) - tr(B)$
- $tr(AB) = tr(BA)$
- $tr(A) = tr(A')$ or $tr(A^T)$
- $tr(I_n) = n$
- $tr(O) = 0$
- $tr(AB) \neq tr\ A \cdot tr\ B$

ADDITION AND SUBTRACTION OF MATRICES

If $A = [a_{ij}]_{m\times n}$ and $B = [a_{ij}]_{m\times n}$ are two matrices of the same order then their sum $A + B$ is a matrix whose each element is the sum of corresponding elements *i.e.*, $A = [a_{ij} + b_{ij}]_{m\times n}$.

Similarly, their subtraction $A - B$ is defined as $A - B = [a_{ij} - b_{ij}]_{m\times n}$

Note : Matrix addition and subtraction can be possible only when matrices are of the same order.

Properties of matrix addition

If A, B and C are matrices of same order, then

- $A + B = B + A$ (Commutative law)
- $(A + B) + C = A + (B + C)$ (Associative law)
- $A + O = O + A$, where O is zero matrix which is additive identity of the matrix.
- $A + (-A) = O = (-A) + A$, where $(-A)$ is obtained by changing the sign of every element of A, which is additive inverse of the matrix.
- $\left.\begin{matrix} A+B=A+C \\ B+A=C+A \end{matrix}\right\} \Rightarrow B = C$ (Cancellation law)

SCALAR MULTIPLICATION OF MATRICES

Let $A = [a_{ij}]_{m\times n}$ be a matrix and k be a scalar, then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by kA.

Thus, if $A = [a_{ij}]_{m\times n}$, then $kA = Ak = [ka_{ij}]_{m\times n}$.

Properties of scalar multiplication

If A, B are matrices of the same order and λ, μ are any two scalars, then

- $\lambda(A + B) = \lambda A + \lambda B$
- $(\lambda + \mu)A = \lambda A + \mu A$
- $\lambda(\mu A) = (\lambda\mu)A = \mu(\lambda A)$
- $(-\lambda A) = -(\lambda A) = \lambda(-A)$

MULTIPLICATION OF MATRICES

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post multiplier). Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = [a_{i1}a_{i2}\ldots a_{in}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

= (i^{th} row of A)(j^{th} column of B) ...(i)

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$

Now we define the product of a row matrix and a column matrix.

Let $A = [a_1\ a_2 \ldots\ldots\ldots\ a_n]$ be a row matrix and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be a column matrix.

Then $AB = [a_1b_1 + a_2b_2 + \ldots.. + a_nb_n]$

Thus, from (i), $(AB)_{ij}$ = Sum of the product of elements of i^{th} row A with the corresponding elements of j^{th} column of B.

Properties of matrix multiplication

If A,B and C are three matrices such that their product is defined, then

- $AB \neq BA$ (Commutative law)
- $(AB)C = A(BC)$ (Associative Law)
- $IA = A = AI$, where I is identity matrix of same order
- $A(B + C) = AB + AC$ (Distributive law)
- If $AB = AC \not\Rightarrow B = C$ (Cancellation law is not applicable)
- If $AB = 0$, it does not mean that $A = 0$ or $B = 0$, again product of two non zero matrix may be a zero matrix.

POSITIVE INTEGRAL POWERS OF A MATRIX

The positive integral powers of a matrix A are defined only when A is a square matrix.

Also then $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A = A^2A$

Also for any positive integers m and n,

- $A^mA^n = A^{m+n}$
- $(A^m)^n = A^{mn} = (A^n)^m$
- $I^n = I$, $I^m = I$
- $A^0 = I_n$, where A is a square matrix of order n.

TRANSPOSE OF A MATRIX

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of matrix A and is denoted by A^T or A'.

Properties of transpose

Let A and B be two matrices of same order $m \times n$ and A^T, B^T be their transpose respectively, then,

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$, k be any scalar (real or complex).
- $(AB)^T = B^TA^T$, A and B being conformable for the product AB.
- $(A_1A_2A_3 \ldots\ldots A_{n-1}A_n)^T = A_n^T A_{n-1}^T \ldots A_3^T A_2^T A_1^T$
- $I^T = I$

SPECIAL TYPES OF MATRICES

(1) Symmetric matrix : A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all i, j or $A^T = A$.

(2) Skew-symmetric matrix : A square matrix $A = [a_{ij}]$ is called skew- symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$.

All principal diagonal elements of a skew- symmetric matrix are always zero because for any diagonal element.
$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

Properties of symmetric and skew-symmetric matrices

- If A is a square matrix, then $A + A^T$, AA^T, A^TA are symmetric matrices, while $A - A^T$ is skew-symmetric matrix.
- If A is a symmetric matrix, then $-A$, kA, A^T, A^n, A^{-1}, B^TAB are also symmetric matrices, where $n \in N$, $k \in R$ and B is a square matrix of order that of A.
- If A is a skew-symmetric matrix, then

(i) A^{2n} is a symmetric matrix for $n \in N$.
(ii) A^{2n+1} is a skew-symmetric matrix for $n \in N$.
(iii) kA is also skew-symmetric matrix, where $k \in R$.
(iv) B^TAB is also skew- symmetric matrix where B is a square matrix of order that of A.

- If A, B are two symmetric matrices, then $A \pm B$, $AB + BA$ are symmetric matrices and $AB - BA$ is a skew- symmetric matrix.
- AB is a symmetric matrix when $AB = BA$.
- If A, B are two skew-symmetric matrices, then $A \pm B$, $AB - BA$ are skew-symmetric matrices and $AB + BA$ is a symmetric matrix.
- If A is a skew-symmetric matrix and C is a column matrix, then C^TAC is a zero matrix.
- Every square matrix A can be uniquely expressed as sum of a symmetric and skew-symmetric matrix

$$i.e.,\ A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

(3) Orthogonal matrix : A square matrix A is called orthogonal if $AA^T = I = A^TA$ *i.e.*, if $A^{-1} = A^T$

(4) Idempotent matrix : A square matrix A is called an idempotent matrix if $A^2 = A$.
In fact, every unit matrix is indempotent.

(5) Involutory matrix : A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$.
Every unit matrix is involutory.

(6) Nilpotent matrix : A square matrix A is called a nilpotent matrix, if there exists p, $p \in N$ such that $A^p = O$.

DETERMINANTS

To every square matrix $A = [a_{ij}]_{m \times n}$ is associated a number of function called the determinant of A and denoted by $|A|$.

Consider three homogeneous linear equations
$a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$
and $a_3x + b_3y + c_3z = 0$

Eliminating x, y, z from above three equations, we obtain
$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$ ...(i)

So, (i) can be also represented by $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

$$i.e.,\ a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

As, it contains three rows and three columns, it is called a determinant of third order.

Properties of determinants

- The value of determinant remains unchanged, if the rows and the columns are interchanged.
- If each element of any row (or column) of determinant can be expressed as a sum of two terms, then the determinant can also be expressed as the sum of two determinants.
- If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.
- If all the elements of any row (or column) be multiplied by the same number, then the value of determinant is multiplied by that number.
- If a determinant has two rows (or columns) identical, then its value is zero.
- The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

- If any row (or column) of determinant is multiplied by a non-zero number, then the determinant is also divided by that number.
- If a determinant D becomes zero on putting $x = \alpha$, then we say that $(x - \alpha)$ is factor of determinant.

PRODUCT OF TWO DETERMINANTS

Let the two determinants of third order be,

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Let D be their product.

$$\text{Then, } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

SINGULAR AND NON-SINGULAR MATRIX

Any square matrix A is said to be non-singular if $|A| \neq 0$, and a square matrix A is said to be singular if $|A| = 0$.

CONJUGATE OF A MATRIX

The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\bar{A}$.

Properties of conjugates

- $\overline{(\bar{A})} = A$
- $\overline{(A + B)} = \bar{A} + \bar{B}$
- $\overline{(\alpha A)} = \bar{\alpha}\bar{A}, \alpha$ being any number
- $\overline{(AB)} = \bar{A}\,\bar{B}, A$ and B being conformable for multiplication.

MINOR OF AN ELEMENT

If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}.

COFACTOR OF AN ELEMENT

The cofactor of an element a_{ij} (*i.e.* the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij}

$C_{ij} = (-1)^{i+j} M_{ij}$

ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.

Thus, adj $A = [C_{ij}]^T$

$\Rightarrow$ (adj $A)_{ij} = C_{ji}$ = cofactor of a_{ji} in A.

Properties of adjoint matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

- $A(\text{adj } A) = |A| \;\; I_n = (\text{adj } A)A$
- $|\text{adj } A| = |A|^{n-1}$
- $\text{adj}(\text{adj } A) = |A|^{n-2}A$
- $|\text{adj}(\text{adj } A) = |A|^{(n-1)^2}$
- $\text{adj}(A^T) = (\text{adj } A)^T$
- $\text{adj }(AB) = (\text{adj } B)(\text{adj } A)$
- $\text{adj}(A^m) = (\text{adj}A)^m, m \in N$
- $\text{adj}(kA) = k^{n-1}(\text{adj}A), k \in R$
- $\text{adj}(I_n) = I_n$

INVERSE OF A MATRIX

A non-singular square matrix A of order n is invertible, if there exists a square matrix B of the same order such that $AB = I_n = BA$.

In such a case, we say that the inverse of A is B and we write $A^{-1} = B$. The inverse of A is given by

$$A^{-1} = \frac{1}{|A|}.\text{adj} A$$

The necessary and sufficient condition for the existence of the inverse of a square matrix A is that $|A| \neq 0$.

Properties of inverse matrix

If A and B are invertible matrices of the same order, then

- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^k)^{-1} = (A^{-1})^k, k \in N$
- $\text{adj}(A^{-1}) = (\text{adj}A)^{-1}$
- $|A^{-1}| = \dfrac{1}{|A|}$
- Every invertible matrix possesses a unique inverse.

RANK OF A MATRIX

The rank of a given matrix A is said to be r if

(i) Every minor of A of order $r + 1$ is zero.

(ii) There is atleast one minor of A of order r which does not vanish.

(iii) The rank r of matrix A is written as $\rho(A) = r$.

Rank of a matrix in Echelon form : The rank of a matrix in Echelon form is equal to the number of non-zero rows in that matrix.

HOMOGENEOUS AND NON-HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

A system of equations $AX = B$ is called a homogeneous system if $B = O$. If $B \neq O$, then it is called a non-homogeneous system of equations.

SOLUTION OF NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

Matrix method

If $AX = B$, then $X = A^{-1}B$ gives a unique solution, provided A is non-singular (*i.e.*, $|A| \neq 0$).

But if $|A| = 0$ and (adj A). $B = 0$, then system is consistent with infinitely many solutions otherwise system is incositent.

SOLUTION OF A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

- Let $AX = O$ be a homogeneous system of three linear equations in three unknowns.
- Write the given system of equations in the form $AX = O$ and write A.
- Find $|A|$.
- If $|A| \neq 0$, then the system is consistent and $x = y = z = 0$ is the unique solution.
- If $|A| = 0$, then the systems of equations has infinitely many solutions. In order to find these solution put $z = k$ (any real number) and solve any two equations for x and y by matrix method. The values of x and y so obtained with $z = k$ give a solution of the given system of equations.

SOLUTION OF SYSTEM OF LINEAR EQUATIONS IN THREE VARIABLES BY CRAMER'S RULE

- The solution of the system of linear equation given by $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

  is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$,

  where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

  $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

  provided that $D \neq 0$

Conditions for consistency

- If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$
- If $D = 0$ and atleast one of the determinants D_1, D_2, D_3 is non-zero, then given system of equations is inconsistent.

PROBLEMS

Single Correct Answer Type

1. If $a \neq b \neq c$, then the value of x which satisfies the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$, is

(a) $x = 0$ (b) $x = a$ (c) $x = b$ (d) $x = c$

2. If $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$, then $x =$

(a) 1, 9 (b) – 1, 9 (c) – 1, – 9 (d) 1, – 9

3. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} =$

(a) $-2abc$ (b) abc
(c) 0 (d) $a^2 + b^2 + c^2$

4. If $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = k(x+y+z)(x-z)^2$, then $k =$

(a) $2xyz$ (b) 1 (c) xyz (d) $x^2y^2z^2$

5. Value of $\Delta = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ equals

(a) $a^3 + b^3 + c^3 - 3abc$
(b) $3abc - a^3 - b^3 - c^3$
(c) $a^3 + b^3 + c^3 - a^2b - b^2c - c^2a$
(d) $(a + b + c)(a^2 + b^2 + c^2 + ab + bc + ca)$

6. If ω is a complex cube root of unity, then the value of $\begin{vmatrix} 2 & 2\omega & -\omega^2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} =$

(a) 0 (b) 1
(c) – 1 (d) None of these

7. If ω be a complex cube root of unity, then

$$\begin{vmatrix} 1 & \omega & -\omega^2/2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} =$$

(a) 0 (b) 1 (c) ω (d) ω^2

8. $\begin{vmatrix} a_1 & ma_1 & b_1 \\ a_2 & ma_2 & b_2 \\ a_3 & ma_3 & b_3 \end{vmatrix} =$

(a) 0 (b) $ma_1a_2a_3$ (c) $ma_1a_2b_3$ (d) $mb_1a_2a_3$

9. $\begin{vmatrix} 11 & 12 & 13 \\ 12 & 13 & 14 \\ 13 & 14 & 15 \end{vmatrix} =$

(a) 1 (b) 0 (c) –1 (d) 67

10. A root of the equation $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$ is

(a) 6 (b) 3
(c) 0 (d) None of these

11. The roots of the equation

$$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0 \text{ are}$$

(a) 1, 2 (b) – 1, 2 (c) 1, – 2 (d) –1, – 2

12. The roots of the determinant equation (in x)

$$\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0$$

(a) $x = a, b$ (b) $x = -a, -b$
(c) $x = -a, b$ (d) $x = a, -b$

13. If a, b, c are in A.P., then the value of

$$\begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix} \text{ is}$$

(a) $x - (a + b + c)$ (b) $9x^2 + a + b + c$
(c) $a + b + c$ (d) 0

14. If $a\ (\neq 6), b, c$ satisfy $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$, then $abc =$

(a) $a + b + c$ (b) 0
(c) b^3 (d) $ab + bc$

15. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 12 \end{bmatrix}$, then

(a) $AB = O, BA = O$ (b) $AB = O, BA \neq O$
(c) $AB \neq O, BA = O$ (d) $AB \neq O, BA \neq O$

16. If ω is a complex cube root of unity, then

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix} =$$

(a) $3\sqrt{3}i$ (b) $-2\sqrt{3}i$
(c) $i\sqrt{3}$ (d) 3

17. $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix} =$

(a) 0 (b) $2abc$
(c) $a^2b^2c^2$ (d) None of these

18. If $\begin{vmatrix} x+1 & 1 & 1 \\ 2 & x+2 & 2 \\ 3 & 3 & x+3 \end{vmatrix} = 0$, then x is

(a) 0, – 6 (b) 0, 6
(c) 6 (d) None of these

19. The roots of the equation $\begin{vmatrix} x & 0 & 8 \\ 4 & 1 & 3 \\ 2 & 0 & x \end{vmatrix} = 0$ are equal to

(a) –4, 4 (b) 2, –4 (c) 2, 4 (d) 2, 8

20. If $\begin{vmatrix} 5 & 3 & -1 \\ -7 & x & -3 \\ 9 & 6 & -2 \end{vmatrix} = 0$, then x is equal to

(a) 3 (b) 5 (c) 7 (d) 9

21. A, B are n-rowed square matrices such that $AB = O$ and B is non-singular. Then

(a) $A \neq O$ (b) $A = O$
(c) $A = I$ (d) None of these

22. The solutions of the equation $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$ are

(a) 3, –1 (b) –3, 1 (c) 3, 1 (d) –3, –1

23. If $f(x)=\begin{vmatrix} x-3 & 2x^2-18 & 3x^3-81 \\ x-5 & 2x^2-50 & 4x^3-500 \\ 1 & 2 & 3 \end{vmatrix}$

then $f(1)\cdot f(3)+f(3)\cdot f(5)+f(5)\cdot f(1)=$

(a) $f(1)$ (b) $f(3)$

(c) $f(1)+f(3)$ (d) $f(1)+f(5)$

24. The value of k for which the set of equations $x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 3y - 4z = 0$ has a non-trivial solution is

(a) 15 (b) 31/2

(c) 16 (d) 33/2

25. If the system of equations, $x + 2y - 3z = 1$, $(k + 3)z = 3$, $(2k + 1)x + z = 0$ is inconsistent, then the value of k is

(a) -3 (b) 1/2 (c) 0 (d) 2

26. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if

(a) $k \neq 0$ (b) $-1 < k < 1$

(c) $-2 < k < 2$ (d) $k = 0$

27. The system of equations $x + y + z = 2$, $3x - y + 2z = 6$ and $3x + y + z = -18$ has

(a) a unique solution

(b) no solutions

(c) an infinite number of solutions

(d) zero solution as the only solution

28. The system of equations

$\alpha x + y + z = \alpha - 1$

$x + \alpha y + z = \alpha - 1$

$x + y + \alpha z = \alpha - 1$

has no solution, if α is

(a) not equal to -2 (b) 1

(c) -2 (d) either -2 or 1

29. If A is a square matrix of order n and $A = kB$, where k is a scalar, then $|A|=$

(a) $|B|$ (b) $k|B|$

(c) $k^n|B|$ (d) $n|B|$

30. If $A = [a\ b]$, $B = [-b\ -a]$ and $C=\begin{bmatrix} a \\ -a \end{bmatrix}$, then the correct statement is

(a) $A = -B$ (b) $A + B = A - B$

(c) $AC = BC$ (d) $CA = CB$

Assertion & Reason Type

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement–2 is true, Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

31. **Statement-1:** The determinants

$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ are not identical.

Statement-2: The first two columns in both the determinants are identical and third column is different.

32. **Statement-1 :** If $A=\begin{bmatrix} 2 & 1+2i \\ 1-2i & 7 \end{bmatrix}$, then $\det(A)$ is real.

Statement-2 : If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, a_{ij} being complex numbers when $i \neq j$, then $\det(A)$ is always real.

33. **Statement-1 :** If A is a matrix of order 2×2, then $|\text{adj}A| = |A|$.

Statement-2 : $|A| = |A^T|$.

34. **Statement-1 :** If $a_1, a_2, \ldots\ldots., a_n, \ldots$ are in G.P. ($a_i > 0$ for all i), then $\Delta=\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}=0$

Statement-2 : The three elements in any row of the determinant are in H.P.

35. **Statement-1 :** If $A=\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, then $|A|=0$

Statement-2 : The value of the determinant of a skew symmetric matrix is always zero.

36. **Statement-1 :** The inverse of the matrix

$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ does not exist.

Statement-2 : If determinant of matrix is zero, then the inverse of that matrix does not exist.

Comprehension Type

Paragraph for Q. No. 37-39

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$. If U_1, U_2 and U_3 are column matrices satisfying

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ and } AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, U \text{ is } 3 \times 3 \text{ matrix}$$

whose columns are U_1, U_2, U_3. Then,

37. The value of $|U|$ is

(a) 3 (b) –3 (c) 3/2 (d) 2

38. The sum of the elements of U^{-1} is

(a) –1 (b) 0 (c) 1 (d) 3

39. The value of $[3\ 2\ 0]\ U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is

(a) [5] (b) $\begin{bmatrix} \frac{5}{2} \end{bmatrix}$ (c) [4] (d) $\begin{bmatrix} \frac{3}{2} \end{bmatrix}$

Matrix-Match Type

40. Match the following :

	Column-I		Column-II
A.	Let $\lvert A \rvert = \lvert a_{ij} \rvert_{3 \times 3} \neq 0$. Each element a_{ij} is multiplied by k^{i-j}. Let $\lvert B \rvert$ the resulting determinant, where $k_1\lvert A \rvert + k_2 \lvert B \rvert = 0$. Then $k_1 + k_2 =$	p.	0
B.	The maximum value of a third order determinant each of its entries are ± 1 equals	q.	4
C.	$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$	r.	1
D.	$\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B$ then $A + 2B$ is	s.	$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

SOLUTIONS

1. (a) : On putting $x = 0$, the determinant (Δ) becomes,

$$\Delta_{x=0} = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = a(bc) - b(ac) = 0$$

$\therefore$ $x = 0$ is a root of the given equation.

2. (d) : We have, $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$

Apply $C_1 \to C_1 + C_2 + C_3$, we get

$$(9+x)\begin{vmatrix} 1 & 3 & 5 \\ 1 & x+2 & 5 \\ 1 & 3 & x+4 \end{vmatrix} = 0$$

Apply $R_1 \to R_1 - R_2$, we get

$$(x+9)\begin{vmatrix} 0 & 1-x & 0 \\ 1 & x+2 & 5 \\ 1 & 3 & x+4 \end{vmatrix} = 0$$

Apply $R_2 \to R_2 - R_3$, we get

$$\Rightarrow (x+9)\begin{vmatrix} 0 & 1-x & 0 \\ 0 & -(1-x) & 1-x \\ 1 & 3 & x+4 \end{vmatrix} = 0$$

$$\Rightarrow (x+9)(1-x)\begin{vmatrix} 0 & 1 & 0 \\ 0 & -1 & 1-x \\ 1 & 3 & x+4 \end{vmatrix} = 0$$

$\Rightarrow (x + 9)(1 - x)(1 - x) = 0$

$\Rightarrow x = 1, 1, -9$

3. (c) : $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$

(Since value of determinant of skew-symmetric matrix of odd order is 0).

4. (b) : $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)\begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$

[Using $R_1 \to R_1 + R_2 + R_3$]

$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & z & x \\ x & y & z \end{vmatrix}$; [Using $C_1 \to C_1 - C_2$]

$= (x + y + z)\{(z^2 - xy) - (xz - x^2) + (xy - xz)\}$
$= (x + y + z)(x - z)^2$
Compare with given condition, we get $k = 1$.

5. (b) : $\Delta = \begin{vmatrix} 2(a+b+c) & 0 & a+b+c \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

[Using $R_1 \to R_1 + R_2 + R_3$]

$\Delta = (a+b+c)\begin{vmatrix} 2 & 0 & 1 \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

On expanding, $\Delta = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
$= -(a^3 + b^3 + c^3 - 3abc) = 3abc - a^3 - b^3 - c^3$.

6. (a) : Let $\Delta \equiv \begin{vmatrix} 2 & 2\omega & -\omega^2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 2+2\omega+2\omega^2 & 2\omega & -\omega^2 \\ 1+1-2 & 1 & 1 \\ 1-1-0 & -1 & 0 \end{vmatrix}$

[Applying $C_1 \to C_1 + C_2 - 2C_3$].

$= \begin{vmatrix} 0 & 2\omega & -\omega^2 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 0$ [$\because 1 + \omega + \omega^2 = 0$]

7. (a) : $\begin{vmatrix} 1 & \omega & -\omega^2/2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix}$

$= -\frac{1}{2}\begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{vmatrix}$ (Applying $C_1 \to C_1 + C_2 + C_3$) $= 0$

8. (a) : $\begin{vmatrix} a_1 & ma_1 & b_1 \\ a_2 & ma_2 & b_2 \\ a_3 & ma_3 & b_3 \end{vmatrix} = m\begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = 0$

$\{\because C_1 \equiv C_2\}$

9. (b) : Let $\Delta = \begin{vmatrix} 11 & 12 & 13 \\ 12 & 13 & 14 \\ 13 & 14 & 15 \end{vmatrix}$

Apply $C_3 \to C_3 - C_2$, we get

$\Delta = \begin{vmatrix} 11 & 12 & 1 \\ 12 & 13 & 1 \\ 13 & 14 & 1 \end{vmatrix}$

Apply $C_2 \to C_2 - C_1$, we get

$\Delta = \begin{vmatrix} 11 & 1 & 1 \\ 12 & 1 & 1 \\ 13 & 1 & 1 \end{vmatrix} = 0$ [$\because C_2 \equiv C_3$]

10. (c) : Applying $C_1 \to C_1 + C_2 + C_3$, we obtain

$-x\begin{vmatrix} 1 & -6 & 3 \\ 1 & 3-x & 3 \\ 1 & 3 & -6-x \end{vmatrix} = 0$

Applying $R_1 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$, we get

$\Rightarrow -x\begin{vmatrix} 1 & -6 & 3 \\ 0 & 9-x & 0 \\ 0 & 9 & -9-x \end{vmatrix} = 0$

$\Rightarrow -x(9 - x)(-9 - x) = 0 \Rightarrow x = 0, 9, -9$.

11. (b) : We have, $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-2 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = 0$

$\Rightarrow (x + 1)(x - 2)^2 = 0 \Rightarrow x = -1, 2$.

12. (a) : Obviously, the determinant is satisfied for $x = a, b$.

13. (d) : Let $A = \begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$, we get

$$A = \begin{vmatrix} x+2 & 1 & x+a \\ x+4 & 1 & x+b \\ x+6 & 1 & x+c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \quad A = \begin{vmatrix} x+2 & 1 & x+a \\ 2 & 0 & b-a \\ 4 & 0 & c-a \end{vmatrix}$$

$= -1\ (2c - 2a - 4b + 4a) = 2(2b - c - a)$

Since, a, b, c are in A.P. $\Rightarrow A = 0$.

14. (c) : We have, $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a-6 & 0 & 0 \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$$

[Applying $R_1 \to R_1 - 2R_2$]

$\Rightarrow \quad (a - 6)(b^2 - ac) = 0 \ \Rightarrow \ b^2 - ac = 0 \ (\because \ a \neq 6)$

$\therefore \quad ac = b^2 \ \Rightarrow \ abc = b^3$.

15. (b) : $AB = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

while $BA = \begin{bmatrix} 0 & 0 \\ 1 & 12 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 25 & 0 \end{bmatrix} \neq O$

16. (a) : $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix} = 3(\omega - \omega^2)$

$$= 3\left[\frac{-1+\sqrt{3}i}{2} - \frac{-1-\sqrt{3}i}{2}\right] = 3\sqrt{3}\,i$$

17. (a) : Put $x = 0$, in given determinant, we get

$$\begin{vmatrix} 4 & 0 & 1 \\ 4 & 0 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 0$$

18. (a)

19. (a) : Let $\Delta = \begin{vmatrix} x & 0 & 8 \\ 4 & 1 & 3 \\ 2 & 0 & x \end{vmatrix}$

$\therefore \quad \Delta = x(x - 0) - 0(4x - 6) + 8(0 - 2)$

or $\quad x^2 - 16 = 0$

$\Rightarrow \ x = 4, -4$.

20. (d) : Given, $\begin{vmatrix} 5 & 3 & -1 \\ -7 & x & -3 \\ 9 & 6 & -2 \end{vmatrix} = 0$

$\Rightarrow \quad 5(-2x + 18) - 3(14 + 27) - 1(-42 - 9x) = 0$

$\therefore \quad -10x + 90 - 42 - 81 + 42 + 9x = 0 \ \Rightarrow \ x = 9$.

21. (b) : Since $|B| \neq 0 \Rightarrow \ B^{-1}$exists.

Now, $AB = O$

$\Rightarrow \ (AB)B^{-1} = OB^{-1} \ \Rightarrow \ A(BB^{-1}) = O$

$\Rightarrow \ AI = O \ \Rightarrow \ A = O$

So, AB and BA are defined only.

22. (a) : $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$

$\Rightarrow \ x(5x - 2x) - 2\ (2x + x) - 1\ (4 + 5) = 0$

$\Rightarrow \ 3x^2 - 6x - 9 = 0$

or $\ x^2 - 2x - 3 = 0$

$\Rightarrow \ (x + 1)(x - 3) = 0 \quad$ or $\quad x = -1, 3$.

23. (b) : Taking out $(x - 3)$, $(x - 5)$ and 2 from I row, II row and II column respectively, we get

$$f(x) = 2(x-3)(x-5)\begin{vmatrix} 1 & x+3 & 3(x^2+3x+9) \\ 1 & x+5 & 4(x^2+5x+25) \\ 1 & 1 & 3 \end{vmatrix}$$

$$\Rightarrow f(x) = 2(x-3)(x-5)\begin{vmatrix} 0 & x+2 & 3(x^2+3x+8) \\ 0 & 2 & x^2+11x+73 \\ 1 & 1 & 3 \end{vmatrix},$$

Applying $R_2 \to R_2 - R_1$

$$= 2(x-3)(x-5)\begin{vmatrix} 1 & x+3 & 3(x^2+3x+9) \\ 0 & 2 & x^2+11x+73 \\ 1 & 1 & 3 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$, we get

$= 2(x - 3)(x - 5)[1(x + 2)(x^2 + 11x + 73) - 6(x^2 + 3x + 8)]$

$= 2(x^2 - 8x + 15)(x^3 + 13x^2 + 95x + 146 - 6x^2 - 18x - 48)$

$= 2(x^2 - 8x + 15)(x^3 + 7x^2 + 77x + 98)$

$= 2(x^5 - x^4 + 36x^3 - 413x^2 + 371x + 1470)$

$f(1) = 2928, f(3) = 0, f(5) = 0$

$\therefore \quad f(1)\cdot f(3) + f(3)\cdot f(5) + f(5)\cdot f(1) = 0 + 0 + 0 = 0 = f(3)$

24. (d) : Given set of equations will have a non-trivial solution, if the determinant of coefficient of x, y, z is zero.

i.e., $\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$

$$\Rightarrow 2k - 33 = 0 \ \Rightarrow k = \frac{33}{2}.$$

25. (a) : For the equation to be inconsistent $D = 0$

$$\therefore\ D=\begin{vmatrix}1 & 2 & -3\\ 0 & 0 & k+3\\ 2k+1 & 0 & 1\end{vmatrix}=0$$

$$\Rightarrow\ k=-3$$

$$\text{and } D_1=\begin{vmatrix}1 & 2 & -3\\ 3 & 0 & 0\\ 0 & 0 & 1\end{vmatrix}\neq 0$$

$\therefore$ System is inconsistent for $k = -3$.

26. (a) : The given system of equations has a unique solution, if $\begin{vmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 3 & 2 & k\end{vmatrix}\neq 0 \Rightarrow k\neq 0$.

27. (a) : Given system of equations can be written as

$$\begin{bmatrix}1 & 1 & 1\\ 3 & -1 & 2\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ 6\\ -18\end{bmatrix}$$

On solving the above system, we get the unique solution as $x = -10$, $y = -4$, $z = 16$.

28. (c) : For no solution or infinitely many solutions,

$$\begin{vmatrix}\alpha & 1 & 1\\ 1 & \alpha & 1\\ 1 & 1 & \alpha\end{vmatrix}=0$$

$\Rightarrow\ \alpha = 1, \alpha = -2$

But for $\alpha = 1$, there are infinitely many solutions and when we put $\alpha = -2$ in given system of equations and add them together, L.H.S $\neq$ R.H.S. *i.e.*, no solution.

29. (c) : $\because A = kB \Rightarrow |A| = k^n|B|$, by fundamental concept.

30. (c) : $AC=[a\ \ b]\begin{bmatrix}a\\ -a\end{bmatrix}=[a^2-ab]$

$$BC=[-b\ \ -a]\begin{bmatrix}a\\ -a\end{bmatrix}=[a^2-ab]$$

$\therefore\ AC = BC$.

31. (d) : The first determinant can be shown to be equal to second.

32. (c) : Clearly, $|A|=\begin{vmatrix}2 & 1+2i\\ 1-2i & 7\end{vmatrix}=9$, which is real.

33. (b) : $\because\ |\text{adj } A|=|A|^{n-1}=|A|$ (Here $n = 2$)

34. (c) : Let r be the common ratio of given G.P.

$$\Delta=\begin{vmatrix}\log a_1+(n-1)\log r & \log a_1+(n)\log r & \log a_1+(n+1)\log r\\ \log a_1+(n+2)\log r & \log a_1+(n+3)\log r & \log a_1+(n+4)\log r\\ \log a_1+(n+5)\log r & \log a_1+(n+6)\log r & \log a_1+(n+7)\log r\end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$=\begin{vmatrix}\log a_1+(n-1)\log r & \log a_1+n\log r & \log a_1+(n+1)\log r\\ \log a_1+(n+2)\log r & \log a_1+(n+3)\log r & \log a_1+(n+4)\log r\\ 3\log r & 3\log r & 3\log r\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, we get

$$=\begin{vmatrix}\log a_1+(n-1)\log r & \log a_1+n\log r & \log a_1+(n+1)\log r\\ 3\log r & 3\log r & 3\log r\\ 3\log r & 3\log r & 3\log r\end{vmatrix}=0$$

35. (c) : The value of the determinant of skew - symmetric matrix of odd order is zero.

36. (a) : Since, det $A = 0$, so inverse of the matrix does not exist.

37. (a) : Let $U=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$$\text{Given, } AU_1=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ 2x+y\\ 3x+2y+z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\qquad \text{...(i)}$$

From (i), we get

$x = 1$ ...(ii), $2x + y = 0$ ...(iii), $(3x + 2y + z)$ ...(iv)

Using (ii), (iii) and (iv), we get $x = 1$, $y = -2$, $z = 1$

$$\therefore\ U_1=\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$$

$$\text{Now, } AU_2=\begin{bmatrix}2\\ 3\\ 0\end{bmatrix}\Rightarrow U_2=\begin{bmatrix}2\\ -1\\ -4\end{bmatrix}$$

$$\text{and } AU_3=\begin{bmatrix}2\\ 3\\ 1\end{bmatrix}\Rightarrow U_3=\begin{bmatrix}2\\ -1\\ -3\end{bmatrix}$$

$$\text{Hence, } U=\begin{bmatrix}1 & 2 & 2\\ -2 & -1 & -1\\ 1 & -4 & -3\end{bmatrix}$$

$\therefore\ |U| = 1(3 - 4) - 2(6 + 1) + 2(8 + 1) = -1 - 14 + 18 = 3$

38. (b) : adj $U=\begin{bmatrix}-1 & -2 & 0\\ -7 & -5 & -3\\ 9 & 6 & 3\end{bmatrix}$ and $|U|=3$

$\therefore\quad U^{-1}=\dfrac{\text{adj}U}{|U|}=\dfrac{1}{3}\begin{bmatrix}-1 & -2 & 0\\ -7 & -5 & -3\\ 9 & 6 & 3\end{bmatrix}$

$\therefore$ Sum of elements of $U^{-1}=0$

39. (a) :

$$\begin{bmatrix}3 & 2 & 0\end{bmatrix}U\begin{bmatrix}3\\2\\0\end{bmatrix}=\begin{bmatrix}3 & 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ -2 & -1 & -1\\ 1 & -4 & -3\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 4 & 4\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix}=\begin{bmatrix}-3+8\end{bmatrix}=\begin{bmatrix}5\end{bmatrix}$$

40. (a) : (A)-(p, s); (B)-(q); (C)-(r); (D)-(p, s)

(A) $|A|=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\quad |B|=\begin{vmatrix}a_{11} & k^{-1}a_{12} & k^{-2}a_{13}\\ ka_{21} & a_{22} & k^{-1}a_{23}\\ k^2a_{31} & ka_{32} & a_{33}\end{vmatrix}$

$$=\frac{1}{k^3}\begin{vmatrix}k^2a_{11} & ka_{12} & a_{13}\\ k^2a_{21} & ka_{22} & a_{23}\\ k^2a_{31} & ka_{32} & a_{33}\end{vmatrix}=|A|$$

$k_1|A|+k_2|B|=0\ \Rightarrow\ k_1+k_2=0$

(B) $\begin{vmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{vmatrix}=4$

(C) $\begin{vmatrix}1 & \cos\alpha & \cos\beta\\ \cos\alpha & 1 & \cos\gamma\\ \cos\beta & \cos\gamma & 1\end{vmatrix}=\begin{vmatrix}0 & \cos\alpha & \cos\beta\\ \cos\alpha & 0 & \cos\gamma\\ \cos\beta & \cos\gamma & 0\end{vmatrix}$

$\Rightarrow\ \sin^2\gamma-\cos\alpha(\cos\alpha-\cos\beta\cos\gamma)$
$\qquad\qquad +\ \cos\beta(\cos\alpha\cos\gamma-\cos\beta)$

$=-\cos\alpha(-\cos\beta\cos\gamma)+\cos\beta(\cos\alpha\cos\gamma)$

$\Rightarrow\ \sin^2\gamma-\cos^2\alpha+2\cos\alpha\cos\beta\cos\gamma-\cos^2\beta$

$=2\cos\alpha\cos\beta\cos\gamma\ \Rightarrow\ \sin^2\gamma=\cos^2\alpha+\cos^2\beta$

$\Rightarrow\ \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

(D) Let $\Delta=\begin{vmatrix}x^2+x & x+1 & x-2\\ 2x^2+3x-1 & 3x & 3x-3\\ x^2+2x+3 & 2x-1 & 2x-1\end{vmatrix}$

Applying $R_2\to R_2-(R_1+R_3)$, we get

$$\Delta=\begin{vmatrix}x^2+x & x+1 & x-2\\ -4 & 0 & 0\\ x^2+2x+3 & 2x-1 & 2x-1\end{vmatrix}=4\begin{vmatrix}x+1 & x-2\\ 2x-1 & 2x-1\end{vmatrix}$$

$$=4\begin{vmatrix}x+1 & -3\\ 2x-1 & 0\end{vmatrix}=24x-12$$

$\therefore\quad A=24,\ B=-12$ So, $A+2B=0$

Top Offbeat Picks Post Class-XII

Marks are not the final decision makers after all.
Here is a list of careers you can pursue even with minimum passing marks in class XII.

◆ INDIAN RAILWAYS INSTITUTE (SCRA)

Special Class Railway Apprentice (SCRA) exam for admission to four-year Mechanical Engineering course at Indian Railways Institute of Mechanical and Electrical Engineering, Jamalpur, is conducted every year by the UPSC. On successful completion of course candidates are posted directly as Group A officer on the post of Assistant Mechanical Engineer (AME) in Indian Railway Service.

Eligibility - Selection would be on basis of entrance exam. Candidates with at least 50% marks in class XII can apply.

Deadline - While the deadline is not out yet, the exam is expected to be held in January.

◆ ENGLISH AND FOREIGN LANGUAGES UNIVERSITY (EFLU)

The EFLU offers BA (Honours) in English and other foreign languages including Arabic, French, German, Japanese, Russian and Spanish. The candidate is not only exposed to linguistics knowledge and literature but also the course opens avenues for diplomatic jobs.

Eligibility - Admissions are granted on the basis of entrance exam. Candidates must clear class XII or its equivalent.

Deadline - Application forms are released around December-January and exam is conducted in February.

◆ INSTITUTE OF COMPANY SECRETARIES OF INDIA (ICSI)

ICSI conducts courses to create Company Secretaries throughout the year. After class XII you can join the eight-month long Foundation Programme. This can be pursued along with graduation studies. Candidates who clear the entrance exam can take admission to Executive Programme.

Eligibility - Candidate must clear the Foundation Programme. Anyone who has cleared class XII is eligible to apply for the same.

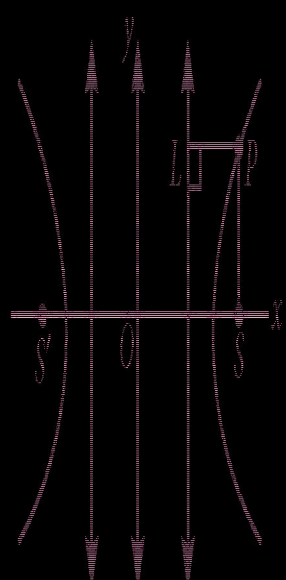
Deadline - Last date to submit application is September 30, 2018.

HYPERBOLA

TANGENTS AND NORMALS

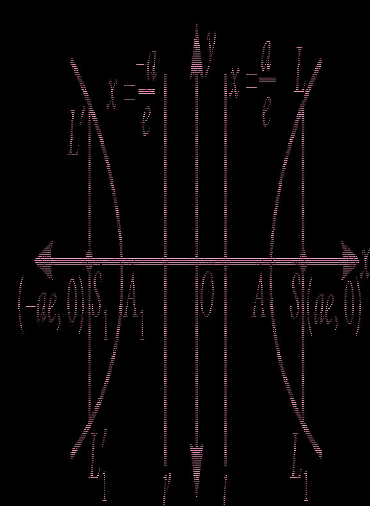
Class XI **Class XII**

- Locus of a point which moves in such a way that the difference of its distance from two fixed points (foci) is always constant.



- General equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a hyperbola if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ \end{vmatrix} \neq 0$ and $h^2 > ab$

Standard form is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Equation of Tangent and Normal

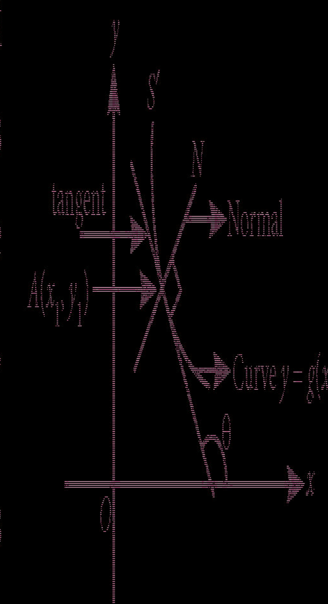
For curve $y = f(x)$

Equation of tangent at $A(x_1, y_1)$ is given by $y - y_1 = m(x - x_1)$

or $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \cdot (x - x_1)$

Equation of normal is given by $y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$

- **Slope of tangent :** Let $y = g(x)$ be a continuous curve & $A(x_1, y_1)$ be any point on the curve, then $\left(\frac{dy}{dx}\right)_{A(x_1, y_1)}$ is



known as the slope of the tangent denoted by tan θ to the curve $y = g(x)$ at point A, where θ is the angle made

CLASS XII Series 4

CBSE DRILL

Synopsis and Chapter wise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2018-19.

Continuity and Differentiability | Application of Derivatives

Continuity and Differentiability

CONTINUITY

- **At a point :** A function $f(x)$ is said to be continuous at a point $x = a$ in the domain of $f(x)$ if all $f(a)$, $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$.
- **On an Open Interval :** A function $f(x)$ is said to be continuous on an open interval (a, b), if it is continuous at each point of (a, b).
- **On a Closed Interval :** A function $f(x)$ is said to be continuous on a closed interval $[a, b]$, if
  - (i) $f(x)$ is continuous from right at $x = a$, *i.e.* $\lim_{h\to 0} f(a+h) = f(a)$
  - (ii) $f(x)$ is continuous from left at $x = b$, *i.e.* $\lim_{h\to 0} f(b-h) = f(b)$
  - (iii) $f(x)$ is continuous at each point of the open interval (a, b).

DISCONTINUITY OF A FUNCTION AND ITS TYPES

1.	**At a point**	A real valued function $f(x)$ is said to be discontinuous at $x = a$, if it is not continuous at $x = a$. The discontinuity may be due to any of the following reasons: (i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but are not equal. This type of discontinuity is non removable of first kind. (ii) $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ or both may not exist. This type of discontinuity is non removable of second kind. (iii) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal but both may not be equal to $f(a)$. It is called removable discontinuity.
2.	**In an interval**	A real valued function $f(x)$ is said to be discontinuous if it is not continuous at atleast one point in the given interval.

Note : If $f(x)$ and $g(x)$ are continuous functions at $x = c$ (real number). Then,

(i) $f + g$ is continuous at $x = c$.

(ii) $af(x)$ is continuous at $x = c$, where a is any real number.

(iii) fg is continuous at $x = c$.

(iv) $\frac{f}{g}$ is continuous at $x = c$, provided $g(c) \neq 0$.

DIFFERENTIABILITY

Let $f(x)$ be a real valued function and a be any real number. Then, we define :

(i) **Right-hand derivative :** $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$, and is denoted by $Rf'(a)$.

(ii) **Left-hand derivative :** $\lim_{h \to 0^-} \frac{f(a-h) - f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$, and is denoted by $Lf'(a)$.

A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.

The common value of $Rf'(a)$ and $Lf'(a)$ is denoted by $f'(a)$ and it is known as the derivative of $f(x)$ at $x = a$.

If, however, $Rf'(a) \neq Lf'(a)$, we say that $f(x)$ is not differentiable at $x = a$.

Note : (i) Every differentiable function is continuous but converse is not true in general.

(ii) A function f is said to be differentiable if it is differentiable at every point in its domain.

Some properties of derivatives

Let $u(x)$ and $v(x)$ be two differentiable functions.

1.	**Sum or Difference Rule**	$(u \pm v)' = u' \pm v'$
2.	**Product Rule**	$(uv)' = u'v + uv'$
3.	**Quotient Rule**	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, v \neq 0$
4.	**Composite Function (Chain Rule)**	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$
5.	**Implicit Function**	Here, we differentiate the function of type $f(x, y) = 0$.
6.	**Logarithmic Function**	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
7.	**Parametric Function**	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, f'(t) \neq 0$
8.	**Second Order Derivative**	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$

Some general derivatives

Function	Derivative	Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	e^{ax}	ae^{ax}	e^x	e^x

$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$; $x \in (-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$; $x \in (-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2}$; $x \in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$; $x \in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$; $x \in R-[-1, 1]$	$\mathrm{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$; $x \in R-[-1,1]$
$\log_e x$	$\frac{1}{x}$; $x > 0$	a^x	$a^x \log_e a$; $a > 0$	$\log_a x$	$\frac{1}{x \log_e a}$; $x > 0$ and $a > 0$

MEAN VALUE THEOREMS

- **Rolle's theorem :** If f: $[a,b] \to R$ is continuous on $[a, b]$, differentiable on (a, b), such that $f(a) = f(b)$, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.
- **Geometrical meaning :** The tangent at point $(c, f(c))$ on the curve $y = f(x)$ is parallel to x-axis.
- **Lagrange's mean value theorem :** If f: $[a, b] \to R$ is continuous on $[a, b]$ and differentiable on (a, b) then there exists at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
- **Geometrical meaning :** The tangent at point $(c, f(c))$ on the curve $y = f(x)$ is parallel to the chord joining $(a, f(a))$ and $(b, f(b))$.

Application of Derivatives

RATE OF CHANGE OF QUANTITIES

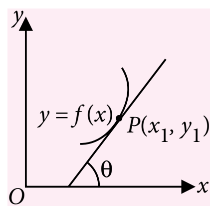
Let $y = f(x)$ be a function. Then dy/dx denotes the rate of change of y w.r.t. x.

Further, $\frac{dy}{dx}$ is positive if y increases as x increases and dy/dx is negative if y decreases as x increases.

INCREASING AND DECREASING FUNCTIONS

	Using definition	Using derivative test
Increasing Function	If $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in (a, b)$ where (a, b) is an open interval contained in domain of f.	If $f'(x) \ge 0$ for each $x \in (a, b)$.
Strictly Increasing Function	If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) > 0$ for each $x \in (a, b)$.
Decreasing Function	If $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) \le 0$ for each $x \in (a, b)$.
Strictly Decreasing Function	If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) < 0$ for each $x \in (a, b)$.

TANGENTS AND NORMALS



- **Slope of a line :** Consider a curve $y = f(x)$ and a point $P(x_1, y_1)$ on this curve as shown. If tangent to the curve at $P(x_1, y_1)$ makes an angle θ with the positive direction of x-axis, then $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \tan\theta = m$ = gradient or slope of tangent to the curve at $P(x_1, y_1)$.

- **Equation of Tangent :** The equation of tangent to the curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

- **Equation of Normal :** Slope of normal at (x_1, y_1)

$$= \frac{-1}{\text{Slope of tangent at } (x_1, y_1)} = \frac{-1}{(dy/dx)_{(x_1, y_1)}}$$

$\therefore$ The equation of normal to the curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

$$y - y_1 = \frac{-1}{(dy/dx)_{(x_1, y_1)}}(x - x_1)$$

Note: (i) If $\frac{dy}{dx} = 0$ at (x_1, y_1), then tangent is parallel to x-axis. In this case, the equation of the tangent at the point (x_1, y_1) is given by $y = y_1$.

(ii) If $\frac{dy}{dx} = \infty$ at (x_1, y_1), then tangent is perpendicular to x-axis. In this case, the equation of the tangent at the point (x_1, y_1) is given by $x = x_1$.

ERROR AND APPROXIMATIONS

Let $y = f(x)$ be a differentiable function and Δx, Δy be the small changes in x and y respectively. Then, we find the approximate value of certain quantity as follows:

(i) Find Δx and x (ii) $\Delta y = \frac{dy}{dx}(\Delta x)$

(iii) $\Delta y = f(x + \Delta x) - f(x)$

MAXIMA AND MINIMA

Maximum value of $f(x)$	Let $f(x)$ be a function with domain $D \subset R$. $f(x)$ is said to attain the maximum value $f(a)$ at point 'a' in D, if $f(x) < f(a)$, for all $x \in D$.
Minimum value of $f(x)$	Let $f(x)$ be a function with domain $D \subset R$. $f(x)$ is said to attain the minimum value $f(a)$ at point 'a' in D, if $f(x) > f(a)$, for all $x \in D$.

- **Critical or stationary point :** The value of x for which $f'(x) = 0$

Methods of finding local maxima and minima

- **First derivative test :** Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then,

Local maxima	If $f'(x)$ changes its sign from + ve to – ve as x increases through c, then c is a point of local maxima.
Local minima	If $f'(x)$ changes its sign from – ve to + ve as x increase through c, then c is a point of local minima.
Point of inflexion	If $f'(x)$ does not changes sign, then c is neither point of local maxima nor a point of local minima. Such a point is called point of inflexion.

- **Second derivative test :** Let f be a function defined on an interval I. Let f be twice differentiable at $c \in I$. Then

(i) If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has local maxima at c and $f(c)$ is local maximum value of $f(x)$.

(ii) If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has local minima at c and $f(c)$ is local minimum value of $f(x)$.

Note: The test fails if $f''(c) = 0$. In this case, we go back to the first derivative test to find whether c is a point of maxima, local minima or a point of inflexion.

- **Absolute maxima and Absolute minima of a function :** Let f be a differentiable function on a closed interval $[a, b]$, then it attains the absolute maximum (absolute minimum) at stationary points (points where $f'(x) = 0$) or at the end points of the interval $[a, b]$.

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 2%.

2. Prove that $\frac{2}{x} + 5$, $x \in R - \{0\}$ is a strictly decreasing function.

3. Find the normal at the point (1, 1) on the curve $2y + x^2 = 3$.

4. Find $\frac{dy}{dx}$, when $x = a(t + \sin t)$ and $y = a(1 - \cos t)$.

5. A stone is dropped into a quiet lake and waves move in a circle at a speed of 3.5 cm/sec. At the instant when the radius of circular wave is 7.5 cm, how fast is the enclosed area increasing?

SHORT ANSWER TYPE

6. Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the line joining (3, 0) and (4, 1).

7. Verify Lagrange's mean value theorem for $f(x) = x^2 + 2x + 3$, $x \in [4, 6]$.

8. Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$.

9. Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.

10. Show that $f(x) = [x]$ is not differentiable at $x = 1$, where $[\cdot]$ is a greatest integer function.

LONG ANSWER TYPE - I

11. If $y = a\cos(\log x) + b\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

12. Find the points on the curve $4x^2 + 9y^2 = 1$, where the tangents are perpendicular to the line $2y + x = 0$.

13. Find the approximate volume of the metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm respectively.

14. Prove that $\sin x(1 + \cos x)$ has a maximum value for $x = \frac{\pi}{3}$.

15. Show that the function

$$f(x) = \begin{cases} \left(\dfrac{e^{1/x} - 1}{e^{1/x} + 1}\right), & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is discontinuous at $x = 0$.

LONG ANSWER TYPE - II

16. If $y = e^x \sin x^3 + (\tan x)^x$, then find $\frac{dy}{dx}$.

17. Determine the values of a, b, c for which the function f defined by

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{when } x < 0 \\ c, & \text{when } x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}, & \text{when } x < 0 \end{cases}$$

is continuous at $x = 0$.

18. Show that the function $f(x) = \cot^{-1}(\sin x + \cos x)$ is a strictly decreasing function in the interval $\left(0, \frac{\pi}{4}\right)$.

19. Verify Lagrange's mean value theorem for the following functions on the indicated intervals.

(i) $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$

(ii) $f(x) = \log_e x$ on $[1, 2]$

20. Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

SOLUTIONS

1. We have, $V = x^3$

Differentiating w.r.t. x, we get $\frac{dV}{dx} = 3x^2$.

We know that $\delta V = \frac{dV}{dx}\delta x$.

Now, δx = change in side = 2% of $x = 0.02\,x$.

$\therefore$ $\delta V = 3x^2 \times 0.02\,x = 0.06\,x^3$ cubic m.

2. Let $f(x) = \frac{2}{x} + 5,\ x \in R - \{0\}$. ...(i)

Diff. (i) w.r.t. x, we get $f'(x) = -\frac{2}{x^2}$.

Since $x^2 > 0$ for all $x \in R$, $x \neq 0$, therefore, $-\frac{2}{x^2} < 0$ for all $x \in R - \{0\}$

$\Rightarrow$ $f'(x) < 0$ for all $x \in R - \{0\}$

$\therefore$ Given function is strictly decreasing.

3. We have, $2y + x^2 = 3$ ...(i)

Differentiating w.r.t. x, we get

$$2\frac{dy}{dx} + 2x = 0 \quad \Rightarrow \frac{dy}{dx} = -x$$

At $P(1, 1)$, $\frac{dy}{dx} = -1$

$\therefore$ Slope of tangent to the curve (i) at $(1, 1) = -1$

$\Rightarrow$ Slope of normal to the curve (i) at $(1, 1) = 1$

Now, equation of normal at $(1, 1)$ is

$y - 1 = 1(x - 1)$ or $x - y = 0$

4. We have, $x = a(t + \sin t) \Rightarrow \frac{dx}{dt} = a(1 + \cos t)$;

$y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a\sin t$.

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{a\sin t}{a(1+\cos t)}$$

$$= \frac{2a\sin(t/2)\cos(t/2)}{2a\cos^2(t/2)} = \tan\frac{t}{2}$$

5. Let r be the radius and A be area enclosed by the circular wave at any time t, then

$A = \pi r^2$ ...(i)

Given, $\frac{dr}{dt} = 3.5$ cm/sec

From (i), $\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \quad \Rightarrow \frac{dA}{dt} = 2\pi r(3.5) = 7\pi r$

$$\Rightarrow \left(\frac{dA}{dt}\right)_{r=7.5} = 7\pi\,(7.5) = 52.5\,\pi \text{ cm}^2\text{/sec.}$$

Hence the enclosed area is increasing at the rate 52.5π cm^2/sec when $r = 7.5$ cm.

6. Given curve is $y = (x - 3)^2$ ...(i)

Let $A \equiv (3, 0)$ and $B \equiv (4, 1)$

Slope of $AB = \frac{1-0}{4-3} = 1$ ...(ii)

From (i), $\frac{dy}{dx} = 2\,(x - 3)$ ...(iii)

$\because$ Tangent is parallel to line AB

$\therefore \frac{dy}{dx} = 1$ ...(iv)

From (ii) and (iii), we have

$2\,(x - 3) = 1 \Rightarrow x = \frac{7}{2}$

From (i), when $x = \frac{7}{2}, y = \frac{1}{4}$

Hence, the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)$.

7. Given, $f(x) = x^2 + 2x + 3$ ...(i)

$f(x)$ being a polynomial function is continuous in $[4, 6]$ and derivable in $(4, 6)$. Thus, both the conditions of Lagrange's mean value theorem are satisfied.

$\therefore$ There exists atleast one real number c in $(4, 6)$ such that $f'(c) = \frac{f(6)-f(4)}{6-4}$ ...(ii)

Now, $f(6) = 51, f(4) = 27$.

Differentiating (i) w.r.t. x, we get

$f'(x) = 2x + 2 \quad \Rightarrow f'(c) = 2c + 2.$

$\Rightarrow 2c + 2 = \frac{51-27}{2} = 12$ (From (ii))

$\Rightarrow \quad c = 5$

Hence, Lagrange's mean value theorem is verified for $c = 5$.

8. We have, $f(x) = 2x - |x| = \begin{cases} 2x - x, \text{ if } x \geq 0 \\ 2x - (-x), \text{ if } x < 0 \end{cases}$

$= \begin{cases} x, \text{ if } x \geq 0 \\ 3x, \text{ if } x < 0 \end{cases}$

Here, $f(0) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} 3x = 0$

and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} x = 0$

$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Thus $f(x)$ is continuous at $x = 0$.

9. Let $x = 3$ and $x + \Delta x = 3.02$. Then, $\Delta x = 0.02$.

We have, $y = f(x) = 3x^2 + 5x + 3$

$\therefore \frac{dy}{dx} = 6x + 5$ ...(i)

Also, $f(3) = 45$

Now, $y = f(x) \quad \Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\Rightarrow \Delta y = (6x + 5)\,\Delta x$ [Using (i)]

$\Rightarrow \Delta y = (6 \times 3 + 5)\,(0.02) = 0.46$

$\therefore f(3.02) = f(x + \Delta x) = y + \Delta y = 45 + 0.46 = 45.46$

10. We have, $Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \frac{[1+h]-[1]}{h} = 0 \quad \{\because [1 + h] = 1 \text{ and } [1] = 1\}$

and $Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{[1-h]-[1]}{-h} = \infty$

$\{\because [1 - h] = 0 \text{ and } [1] = 1\}$

Thus, $Rf'(1) \neq Lf'(1)$

Hence, $f(x) = [x]$ is not differentiable at $x = 1$.

11. Given, $y = a\cos(\log x) + b\sin(\log x)$ ...(i)

$\therefore \frac{dy}{dx} = -a\sin(\log x).\frac{1}{x} + b\cos(\log x).\frac{1}{x}$

or $x.\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$

Differentiating again w.r.t. x, we get

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -a\cos(\log x).\frac{1}{x} - b\sin(\log x).\frac{1}{x}$

or $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[a\cos(\log x) + b\sin(\log x)] = -y$

[From (i)]

or $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

12. The equation of the given line is $y = -\frac{1}{2}x$.

$\therefore$ Slope of given line $= -\frac{1}{2}$ ... (i)

Let the required point be (x_1, y_1).

Now, $4x^2 + 9y^2 = 1$ On differentiating w.r.t. x, we get

$\Rightarrow 8x + 18y.\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \left(\frac{-4x}{9y}\right)$

$\therefore$ Slope of the tangent at $(x_1, y_1) = \frac{-4x_1}{9y_1}$ ...(ii)

According to question, $\dfrac{-4x_1}{9y_1}\times\left(-\dfrac{1}{2}\right)=-1$

or $y_1 = -\dfrac{2}{9}x_1$ ...(iii)

Since (x_1, y_1) lies on the curve $4x^2 + 9y^2 = 1$

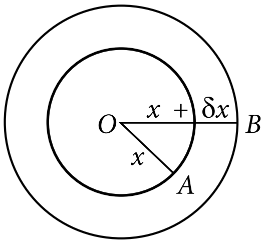
$\therefore\ 4x_1^2 + 9y_1^2 = 1 \Rightarrow 4x_1^2 + 9\times\dfrac{4}{81}x_1^2 = 1$ [Using (iii)]

$\Rightarrow x_1^2 = \dfrac{9}{40}$ or $x_1 = \pm\dfrac{3}{2\sqrt{10}}$.

So, $y_1 = \mp\dfrac{2}{9}\times\dfrac{3}{2\sqrt{10}} = \mp\dfrac{1}{3\sqrt{10}}$

Hence, the required points are $\left(\dfrac{3}{2\sqrt{10}}, \dfrac{-1}{3\sqrt{10}}\right)$ and $\left(\dfrac{-3}{2\sqrt{10}}, \dfrac{1}{3\sqrt{10}}\right)$.

13. Let x be the radius and y the volume of the solid sphere, then



$y = \dfrac{4}{3}\pi x^3$ ...(i)

$\therefore\ \dfrac{dy}{dx} = 4\pi x^2$ ...(ii)

Now, volume of the metal in hollow spherical shell = Change in volume of the solid sphere when its radius increases from 3 cm to 3.0005 cm.

Let $x = 3$ cm and $x + \delta x = 3.0005$ cm

Then, $\delta x = 0.0005$ cm

Now, change in the volume of solid sphere *i.e.*, volume of the hollow spherical shell is given by

$\delta y = \dfrac{dy}{dx}.\delta x = 4\pi x^2\,\delta x = 4\pi(3)^2\,(0.0005) = 0.018\,\pi\ \text{cm}^3$

14. Let $y = f(x) = \sin x\,(1 + \cos x)$ ...(i)

$\therefore\ \dfrac{dy}{dx} = \cos x\,(1 + \cos x) + \sin x\,(-\sin x)$

$= \cos x + (\cos^2 x - \sin^2 x) = \cos x + \cos 2x$ ...(ii)

$\therefore\ \dfrac{d^2y}{dx^2} = -\sin x - 2\sin 2x$ (iii)

At $x = \dfrac{\pi}{3}, \dfrac{dy}{dx} = \cos\dfrac{\pi}{3} + \cos\dfrac{2\pi}{3} = \dfrac{1}{2} - \dfrac{1}{2} = 0$ (Using (ii))

At $x = \dfrac{\pi}{3}, \dfrac{d^2y}{dx^2} = -\sin\dfrac{\pi}{3} - 2\sin\dfrac{2\pi}{3}$

$= -\dfrac{\sqrt{3}}{2} - 2\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{-3\sqrt{3}}{2} < 0$

Hence, $f(x)$ has maximum value at $x = \dfrac{\pi}{3}$

15. We have, $f(x) = \begin{cases}\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right), & \text{when } x \neq 0\\ 0, & \text{when } x = 0\end{cases}$

Clearly, $f(0) = 0$.

Now, R.H.L. $= \lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0 + h)$

$= \lim\limits_{h\to 0} f(h) = \lim\limits_{h\to 0}\left(\dfrac{e^{1/h}-1}{e^{1/h}+1}\right)$

$= \lim\limits_{h\to 0}\dfrac{e^{1/h}\left(1-\dfrac{1}{e^{1/h}}\right)}{e^{1/h}\left(1+\dfrac{1}{e^{1/h}}\right)} = \lim\limits_{h\to 0}\left(\dfrac{1-\dfrac{1}{e^{1/h}}}{1+\dfrac{1}{e^{1/h}}}\right) = 1$

Also, L.H.L. $= \lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0 - h)$

$= \lim\limits_{h\to 0} f(-h) = \lim\limits_{h\to 0}\left(\dfrac{e^{-1/h}-1}{e^{-1/h}+1}\right) = \lim\limits_{h\to 0}\dfrac{\left(\dfrac{1}{e^{1/h}}-1\right)}{\left(\dfrac{1}{e^{1/h}}+1\right)} = -1$

Thus, $\lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$

Hence, $f(x)$ is discontinuous at $x = 0$.

16. Let $u = e^x \sin x^3$ and $v = (\tan x)^x$

Now, $u = e^x \sin x^3$ ...(i)

Differentiating (i) w.r.t. x, we get

$\dfrac{du}{dx} = e^x\cdot\dfrac{d\{\sin(x)^3\}}{dx} + \sin x^3\cdot\dfrac{d}{dx}(e^x)$

$= e^x\cdot\cos x^3\cdot 3x^2 + \sin x^3\cdot e^x$

Hence, $\dfrac{du}{dx} = 3x^2 e^x \cos x^3 + e^x \sin x^3$

Again, $v = (\tan x)^x$

$\Rightarrow \log v = x\log(\tan x)$

Differentiating w.r.t. x, we get

$\dfrac{1}{v}\dfrac{dv}{dx} = 1\cdot\log(\tan x) + x\cdot\dfrac{1}{\tan x}\sec^2 x$

$\therefore\ \dfrac{dv}{dx} = v\,[\log(\tan x) + x\cot x\sec^2 x]$

$\Rightarrow \dfrac{dv}{dx} = (\tan x)^x\,[\log(\tan x) + x\cot x\sec^2 x]$

Now, $y = u + v$

$\therefore\ \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} = 3x^2 e^x\cos(x^3) + e^x\sin(x^3) +$
$(\tan x)^x\,[\log(\tan x) + x\cot x\sec^2 x]$

MPP-4 CLASS XII ANSWER KEY

1. (b)	**2.** (d)	**3.** (b)	**4.** (b)	**5.** (a)
6. (c)	**7.** (b, c)	**8.** (a, b)	**9.** (b, d)	**10.** (c, d)
11. (a, b)	**12.** (a, c)	**13.** (b, c)	**14.** (a)	**15.** (b)
16. (d)	**17.** (2)	**18.** (4)	**19.** (2)	**20.** (7)

17. L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\frac{\sin(a+1)x+\sin x}{x}$

$= \lim_{x\to 0}\left[\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right]$

$= \lim_{x\to 0}\frac{\sin(a+1)x}{(a+1)x}(a+1)+\lim_{x\to 0}\frac{\sin x}{x}$

$= (a + 1) + 1 = a + 2$

R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}}$

$= \lim_{x\to 0}\frac{(\sqrt{x+bx^2}-\sqrt{x})(\sqrt{x+bx^2}+\sqrt{x})}{bx^{\frac{3}{2}}(\sqrt{x+bx^2}+\sqrt{x})}$

$= \lim_{x\to 0}\frac{x+bx^2-x}{bx^{\frac{3}{2}}(\sqrt{x+bx^2}+\sqrt{x})} = \lim_{x\to 0}\frac{bx^2}{bx^2(\sqrt{1+bx}+1)}$

$= \lim_{x\to 0}\frac{1}{\sqrt{1+bx}+1} = \frac{1}{2}, b\neq 0$

Given, $f(0) = c$.

Since $f(x)$ is continuous at $x = 0$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\therefore\ a + 2 = \frac{1}{2} = c \Rightarrow a = -\frac{3}{2}, c = \frac{1}{2}$ and $b \in R - \{0\}$.

18. We have, $f(x) = \cot^{-1}(\sin x + \cos x)$

$\Rightarrow f'(x) = \frac{-1}{1+(\sin x+\cos x)^2}\cdot\frac{d}{dx}(\sin x+\cos x)$

$\Rightarrow f'(x) = \frac{-1}{1+(\sin x+\cos x)^2}\cdot(\cos x-\sin x)$

$$\Rightarrow f'(x) = \frac{-1}{1+2\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)^2}\cdot\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)$$

$$\Rightarrow f'(x) = \frac{-\sqrt{2}\left(\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right)}{1+2\left(\cos\frac{\pi}{4}\sin x+\sin\frac{\pi}{4}\cos x\right)^2}$$

$$\Rightarrow f'(x) = \frac{-\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+2\sin^2\left(x+\frac{\pi}{4}\right)}$$

Since $x \in (0, \pi/4)$, then $0 < x < \frac{\pi}{4}$

$\Rightarrow 0+\frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{4}+\frac{\pi}{4} \Rightarrow \frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{2}$

Since $\sin\left(x+\frac{\pi}{4}\right) > 0$ in $\frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{2}$

and $\cos\left(x+\frac{\pi}{4}\right) > 0$ in $\frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{2}$

$$\Rightarrow \begin{cases} 1+2\sin^2\left(x+\frac{\pi}{4}\right) > 0 \text{ in } \frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{2} \\ \cos\left(x+\frac{\pi}{4}\right) > 0 \text{ in } \frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{2} \end{cases}$$

$$\Rightarrow \left\{ \frac{\cos\left(x+\frac{\pi}{4}\right)}{1+2\sin^2\left(x+\frac{\pi}{4}\right)} > 0 \text{ in } \frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{2} \right.$$

$$\Rightarrow \left\{ \frac{-\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+2\sin^2\left(x+\frac{\pi}{4}\right)} < 0 \text{ in } \frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{2} \right.$$

$\Rightarrow f'(x) < 0$ in $\frac{\pi}{4} < x+\frac{\pi}{4} < \frac{\pi}{2}$ or in $0 < x < \frac{\pi}{4}$

Hence, $f(x)$ is strictly decreasing in $\left(0, \frac{\pi}{4}\right)$.

19. (i) Since $\sin x$ and $\sin 2x$ are everywhere continuous and differentiable, therefore $f(x)$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$. Thus, $f(x)$ satisfies both the conditions of Lagrange's mean value theorem. Consequently, there exists atleast one $c \in (0, \pi)$ such that $f'(c) = \frac{f(\pi)-f(0)}{\pi-0}$. ...(i)

Now, $f(x) = 2\sin x + \sin 2x$

$\Rightarrow f'(x) = 2\cos x + 2\cos 2x$

$\therefore f(0) = 0$ and $f(\pi) = 2\sin\pi + \sin 2\pi = 0$

$\therefore$ (i) becomes,

$2\cos c + 2\cos 2c = \frac{0-0}{\pi-0}$

$\Rightarrow 2\cos c + 2\cos 2c = 0$

$\Rightarrow \cos 2c = -\cos c \Rightarrow \cos 2c = \cos(\pi - c)$

$\Rightarrow 2c = \pi - c \Rightarrow 3c = \pi \Rightarrow c = \pi/3$

Thus, $c = \pi/3 \in (0, \pi)$ such that $f'(c) = \frac{f(\pi)-f(0)}{\pi-0}$.

Hence, Lagrange's mean value theorem is verified.

(ii) Since $f(x) = \log_e x$ is differentiable and so continuous for all $x > 0$. So, $f(x)$ is continuous on $[1, 2]$ and differentiable on $(1, 2)$. Thus, both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (1, 2)$ such that

$f'(c) = \frac{f(2)-f(1)}{2-1}$

Now, $f(x) = \log_e x \Rightarrow f'(x) = \dfrac{1}{x}$

$\Rightarrow$ $f(2) = \log_e 2$ and $f(1) = \log_e 1 = 0$

$\therefore$ $f'(c) = \dfrac{f(2)-f(1)}{2-1}$

$\Rightarrow$ $\dfrac{1}{c} = \dfrac{\log_e 2 - 0}{2-1}$ $\Rightarrow$ $\dfrac{1}{c} = \log_e 2$

$\Rightarrow$ $c = \dfrac{1}{\log_e 2} = \log_2 e$ $\quad\left[\because \log_b a = \dfrac{1}{\log_a b}\right]$

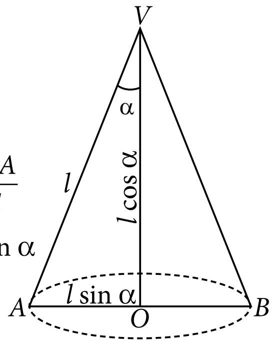
Now, $2 < e < 4 \Rightarrow \log_2 2 < \log_2 e < \log_2 4 \Rightarrow 1 < \log_2 e < 2$.

Thus, $c = \log_2 e \in (1, 2)$ such that $f'(c) = \dfrac{f(2)-f(1)}{2-1}$.

Hence, Lagrange's mean value theorem is verified.

20. Let α be the semi-vertical angle of a cone of given slant height l.

In ΔAOV,



$\cos\alpha = \dfrac{VO}{VA}$ and $\sin\alpha = \dfrac{OA}{VA}$

$\Rightarrow$ $\cos\alpha = \dfrac{VO}{l}$ and $\sin\alpha = \dfrac{OA}{l}$

$\Rightarrow$ $VO = l\cos\alpha$ and $OA = l\sin\alpha$

Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi(OA)^2(VO) = \frac{1}{3}\pi(l\sin\alpha)^2(l\cos\alpha)$$

$$\Rightarrow V = \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha$$

$$\Rightarrow \frac{dV}{d\alpha} = \frac{\pi}{3}l^3(-\sin^3\alpha + 2\sin\alpha\cos^2\alpha)$$

$$\Rightarrow \frac{dV}{d\alpha} = \frac{\pi l^3}{3}\sin\alpha(-\sin^2\alpha + 2\cos^2\alpha) \quad ...(i)$$

The critical points of V are given by $\dfrac{dV}{d\alpha} = 0$.

$$\Rightarrow \frac{\pi l^3}{3}\sin\alpha(-\sin^2\alpha + 2\cos^2\alpha) = 0$$

$\Rightarrow$ $2\cos^2\alpha = \sin^2\alpha$

$\Rightarrow$ $\tan^2\alpha = 2 \Rightarrow \tan\alpha = \sqrt{2}$ $[\because \alpha \text{ is acute} \Rightarrow \sin\alpha \neq 0]$

$$\therefore \cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{3}}$$

Differentiating (i) with respect to α, we get

$$\frac{d^2V}{d\alpha^2} = \frac{\pi l^3}{3}(-3\sin^2\alpha\cos\alpha + 2\cos^3\alpha - 4\sin^2\alpha\cos\alpha)$$

$$= \frac{\pi l^3}{3}\cos^3\alpha\,(2 - 7\tan^2\alpha)$$

$$\therefore \left(\frac{d^2V}{d\alpha^2}\right)_{\tan\alpha=\sqrt{2}} = \frac{1}{3}\pi l^3\left(\frac{1}{\sqrt{3}}\right)^3(2 - 7\times 2)$$

$$= \frac{-4\pi l^3}{3\sqrt{3}} < 0$$

Thus, V is maximum, when $\tan\alpha = \sqrt{2}$ or $\alpha = \tan^{-1}\sqrt{2}$ *i.e.* when the semi-vertical angle of the cone is $\tan^{-1}\sqrt{2}$.

◈◈

National Testing Agency (NTA) To Conduct JEE Main Exam Twice From Next Year

Union education minister Prakash Javadekar has announced that the national level engineering entrance examination JEE Main will be held twice a year from 2019 in online mode. The minister announced this on 7th July. According to the minister, Joint Entrance Examination (JEE) Main will be held by newly formed examination conducting agency, National Testing Agency (NTA). This competitive exam will be held on multiple dates. JEE Main exam will be held in January and April.

Central Board of Secondary Education (CBSE) is currently the nodal agency responsible for organising JEE Main Exam. According to the minister, the candidates who will be appearing for this test will be allowed to use the best score from the examination in counselling process. The minister also said that syllabus, question formats, language and fees for the exam would not be changed. The exam will be more secure and at par with international norms. There will be no issues of leakage and it would be more student friendly, open, scientific and a leak-proof system.

The NTA will benefit the students and they will have the option of going to computer centres from August 2018 to practice for the exams. The tests will be computer-based. The exams will be held on multiple days and students will have the option of choosing the dates. The time table of the exams to be conducted by NTA would be uploaded on the ministry's website.

◈◈

MPP-4 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Continuity and Differentiability

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Let $f : R \to R$ be a function satisfying $f(x + y) = f(x) + \lambda xy + 3x^2y^2$ for all $x, y \in R$.
If $f(3) = 4$ and $f(5) = 52$, then $f'(x)$ is equal to
(a) $10\,x$ (b) $-10\,x$ (c) $20\,x$ (d) $128\,x$

2. If $x = \sin t - \operatorname{cosec} t$, $y = \sin^5 t - \operatorname{cosec}^5 t$, then $(x^2 + 4)\, y'' =$
(a) $xy' - 5y$ (b) $-xy' + 5y$
(c) $xy' - 25y$ (d) $-xy' + 25y$

3. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then
(a) g is not differentiable at $x = 0$
(b) $g'(0) = \cos(\log 2)$
(c) $g'(0) = -\cos(\log 2)$
(d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

4. If $f(x) = \begin{cases} \dfrac{\sin\{\cos x\}}{x - x/2}, & x \neq \dfrac{\pi}{2} \\ 1, & x = \pi/2 \end{cases}$ where $\{\cdot\}$ represents the fractional part function, then $f(x)$ is
(a) continuous at $x = \pi/2$
(b) $\lim\limits_{x \to \pi/2} f(x)$ exists, $f(x)$ is not continuous at $x = \pi/2$
(c) $\lim\limits_{x \to \pi/2^+} f(x)$ does not exist
(d) none of these

5. The derivative of $f(x) = \begin{cases} \dfrac{x - 1}{2x^2 - 7x + 5}, & x \neq 1 \\ -\dfrac{1}{3}, & x = 1 \end{cases}$ at $x = 1$ is
(a) $-\dfrac{2}{9}$ (b) $-\dfrac{2}{44}$
(c) 0 (d) none of these

6. If $f(x) = \left[\dfrac{(x-2)^3}{a}\right] \sin(x - 2) + a \cos(x - 2)$, $[\cdot]$ denotes the greatest integer function, is continuous and differentiable in (4, 6), then
(a) $a \in [8, 64]$ (b) $a \in (0, 8]$
(c) $a \in [64, \infty)$ (d) $a \in (0, 6) \cup (6, 10]$

One or More Than One Option(s) Correct Type

7. If $f(x) = \begin{cases} \dfrac{x}{1 + e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then
(a) $f'(0^+) = 1$ (b) $f'(0^+) = 0$
(c) $f'(0^-) = 1$ (d) $f'(0^-) = 0$

8. If $f(x) = \begin{cases} \dfrac{x}{1 + |x|}, & |x| \geq 1 \\ \dfrac{x}{1 - |x|}, & |x| < 1 \end{cases}$, then which of the following is false?
(a) $f(x)$ is discontinuous and non-differentiable at $x = -1, 1, 0$
(b) $f(x)$ is discontinuous and non-differentiable at $x = -1$, whereas continuous and differentiable at $x = 0, 1$
(c) $f(x)$ is discontinuous and non-differentiable at $x = -1, 1$, whereas continuous and differentiable at $x = 0$.
(d) none of these

9. If the function
$$f(x) = \begin{cases} x + a^2\sqrt{2}\sin x, & 0 \leq x < \pi/4 \\ x\cot x + b, & \pi/4 \leq x < \pi/2 \\ b\sin 2x - a\cos 2x, & \pi/2 \leq x \leq \pi \end{cases}$$
is continuous in the interval $[0, \pi]$, then the ordered pair (a, b) is
(a) $(-1, -1)$ (b) $(0, 0)$
(c) $(-1, 1)$ (d) $(1, 1)$

10. If $f(x) = \frac{x}{2} - 1$, then on the interval $[0, \pi]$

(a) $\tan f(x)$ and $\frac{1}{f(x)}$ are continuous

(b) $\tan f(x)$ and $\frac{1}{f(x)}$ are discontinuous

(c) $\tan f(x)$ and $\frac{1}{f^{-1}(x)}$ are continuous

(d) $\tan f(x)$ is continuous but $\frac{1}{f(x)}$ is not continuous

11. If $f(x) = x\left(\sqrt{x} + \sqrt{x+1}\right)$, then which of the following statements is (are) false?

(a) $f(x)$ is not continuous at $x = 0$

(b) $f(x)$ is continuous at $x = 0$, but not differentiable at $x = 0$

(c) $f(x)$ is differentiable at $x = 0$

(d) none of these

12. Identify the false statement for

$$f(x) = \begin{cases} \frac{x^2}{2} & , \quad 0 \le x < 1 \\ 2x^2 - 3x + \frac{3}{2} & , \quad 1 \le x \le 2 \end{cases}$$

(a) f, f' and f'' are continuous in $[0, 2]$

(b) f and f' are continuous in $[0, 2]$ whereas f'' is continuous in $[0, 1) \cup (1, 2]$

(c) f and f' are differentiable in $[0, 2]$

(d) none of these

13. If $f(x) = \cos\left[\frac{\pi}{x}\right]\cos\left(\frac{\pi}{2}(x-1)\right)$, where $[\cdot]$ denotes the greatest integer function, then $f(x)$ is continuous at

(a) $x = 0$ (b) $x = 1$

(c) $x = 2$ (d) none of these

Comprehension Type

If $f(x + y) = f(x) + f(y)$ for all $x, y \in R$ and $f(1) = 1$ and $g(0) = \lim_{x \to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$. then,

14. The value of $f(x)$ is

(a) x (b) x^2

(c) $3x$ (d) none of these

15. The value of $g(0)$ is

(a) $\log_e 2$ (b) $\frac{1}{2}\log_e 2$

(c) $2\log_e 2$ (d) $\log_e\left(\frac{1}{2}\right)$

Matrix Match Type

16. Match the following:

	Column-I		Column-II
P.	If $f(x) = \begin{cases} x, & x \text{ is rational} \\ x^2, & x \text{ is irrational} \end{cases}$ then the number of points at which $f(x)$ is continuous is	1.	0
Q.	The number of solutions of the equation $[x - 1] + [x + 1] = 0$ is	2.	1
R.	$f(x) = x^2 \sin\left(\frac{1}{x}\right), x \ne 0, f(0) = 0$, then $f'(0^-)$ is	3.	2
S.	$f(x) = \lvert x - 1\rvert + \lvert x\rvert + \lvert x + 1\rvert$, then $f'(0^+)$ is	4.	> 3

	P	Q	R	S
(a)	1	3	4	2
(b)	2	4	3	1
(c)	4	2	3	1
(d)	3	4	1	2

Integer Answer Type

17. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is

18. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ and $kx\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$, then the value of constant k must be

19. If $f(x) = \text{sgn}(x^2 - ax + 1)$ has exactly one point of discontinuity, then the value of $|a|$ equals

20. If Rolle's theorem holds for the function $f(x) = 2x^3 + bx^2 + cx$, $x \in [-1, 1]$, at the point $x = 1/2$, then $2b - 3c$ equals

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SELF CHECK

No. of questions attempted ……

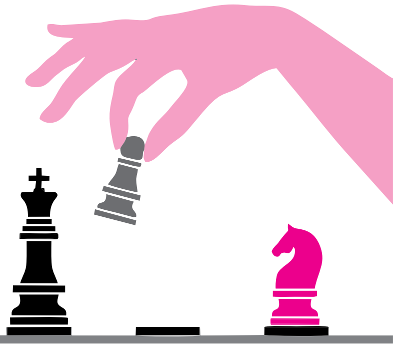
No. of questions correct ……

Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

Challenging PROBLEMS



Differential Calculus

1. $\lim\limits_{n\to\infty}(\sqrt{2}-\sqrt[3]{2})(\sqrt{2}-\sqrt[5]{2})...(\sqrt{2}-\sqrt[2n+1]{2})=$

(a) -1 (b) 0
(c) 1 (d) does not exist

2. $\lim\limits_{n\to\infty}\frac{1}{n}\left[\left(a+\frac{1}{n}\right)^2+\left(a+\frac{2}{n}\right)^2+....+\left(a+\frac{n-1}{n}\right)^2\right]=$

(where $a\in R$)
(a) a^2+a (b) a^2-a
(c) $a^2+a+\frac{1}{3}$ (d) $a^2-a+\frac{1}{3}$

3. $\lim\limits_{n\to\infty}\left(1-\frac{2}{2\cdot 3}\right)\left(1-\frac{2}{3\cdot 4}\right)....\left(1-\frac{2}{(n+1)(n+2)}\right)=$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

4. $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\frac{k^3+6k^2+11k+5}{(k+3)!}=$

(a) $\frac{2}{3}$ (b) $\frac{7}{3}$ (c) $\frac{4}{3}$ (d) $\frac{5}{3}$

5. For $|x|>1$, $\lim\limits_{n\to\infty}\prod_{k=0}^{n}\left(1+\frac{2}{x^{2^k}+x^{-2^k}}\right)=$

(a) $\frac{x+1}{x-1}$ (b) $\frac{x+1}{1-x}$ (c) 0 (d) $+\infty$

6. Value of $\lim\limits_{n\to\infty}\sqrt[n]{2\sin^2\frac{n^{21}}{n+1}+\cos^2\frac{n^{21}}{n+1}}=$

(a) 0 (b) 1
(c) $\sqrt{2}$ (d) does not exist

7. Given $a\in\{1, 2, 3, ..., 9\}$ then

$$\lim_{n\to\infty}\frac{a+aa+...+\underbrace{aaa...a}_{n\text{ digits}}}{10^n}=$$

(a) $\frac{10a}{81}$ (b) $\frac{a}{81}$ (c) $\frac{9a}{10}$ (d) $\frac{a}{10}$

8. $\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+n})=$

(a) 0 (b) 1
(c) $\sqrt{0.5}$ (d) does not exist

9. If $f(x)=\lim\limits_{m\to\infty}\frac{\log(e^m+x^m)}{m}$, $x\ge 0$ then

(a) $f(x)$ is discontinuous on $[0, \infty)$
(b) $f(x)$ is non-diffterentiable on $[0, \infty)$
(c) Area bounded by $f(x)$ between $x = 0$, $x = e$ and x-axis is 2 sq. units
(d) $f(x)$ is a periodic function in $[0, \infty)$

10. $\sum_{k=0}^{2n}(-1)^k\,{}^{2n}C_k\cdot k^n, n\ge 1=$

(a) 0 (b) n^2 (c) $n!$ (d) $\frac{n}{2}$

11. Let f is differentiable at a.

$$\lim_{n\to\infty} n\left[f\left(a+\frac{1}{n}\right)+f\left(a+\frac{2}{n}\right)+...+f\left(a+\frac{k}{n}\right)-kf(a)\right],$$

is ($k\in N$, $a>0$)

(a) $\frac{k(k+1)}{2}f'(a)$ (b) $\frac{k(k-1)}{2}f'(a)$
(c) $\frac{k(k+1)}{2}f(a)$ (d) $\frac{k(k-1)}{2}f(a)$

12. For $m, k\in N$, value of

$$\lim_{n\to\infty}\left[\frac{(n+1)^m+(n+2)^m+....+(n+k)^m}{n^{m-1}}-kn\right]\text{is}$$

(a) $\frac{k(k+1)}{2}m$ (b) $\frac{k(k+1)}{2}$
(c) $\frac{m(m+1)}{2}$ (d) $\frac{k(k-1)}{2}m$

By : Tapas Kr. Yogi, Visakhapatnam Mob : 09533632105

13. Assume that $f(0) = 0$ and f is differentiable at zero. For a positive integer k,

$$\lim_{x\to 0}\frac{1}{x}\cdot\left[f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+....+f\left(\frac{x}{k}\right)\right]=$$

(a) $\left(1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{k}\right)f'(0)$

(b) $\left(1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{k}\right)f''(0)$

(c) $\left(1-\frac{1}{2}+\frac{1}{3}-....+\frac{(-1)^k}{k+1}\right)f'(0)$

(d) $\left(1-\frac{1}{2}+\frac{1}{3}-....+\frac{(-1)^k}{k+1}\right)f''(0)$

14. Let $f:R\to(0,\infty)$ be such that $f(x)+\frac{e^{x+x^2}}{f(x)}\le e^x+e^{x^2}$ $\forall\, x>0$ then $\lim_{x\to 1} f(x)=$

(a) e (b) $\frac{e}{2}$ (c) $2e$ (d) $\sqrt{e}$

15. $\lim_{x\to\pi}\frac{2^{\cot x}+3^{\cot x}-5^{1+\cot x}+2}{2^{2\cot x}+3^{\cot x}-5^{\cot x}+1}=$

(a) -2 (b) 2

(c) 5 (d) does not exist

SOLUTIONS

1. **(b) :** We have,

$$0<(\sqrt{2}-\sqrt[3]{2})(\sqrt{2}-\sqrt[5]{2})...(\sqrt{2}-\sqrt[2n+1]{2})<(\sqrt{2}-1)^n$$

So, limit of the sequence = 0

2. **(c) :** Simplifying given limit, we get

$$\lim_{n\to\infty}\left[\frac{n-1}{n}a^2+\frac{n(n-1)}{n^2}a+\frac{1+2^2+3^2+...+(n-1)^2}{n^3}\right]$$

$$=a^2+a+\frac{1}{3}$$

3. **(c) :** Using, $1-\frac{2}{k(k+1)}=\frac{(k+2)(k-1)}{k(k+1)}$

The given terms simplify to $\frac{1}{3}\cdot\frac{n+3}{n+1}\to\frac{1}{3}$ as $n\to\infty$

4. **(d) :** $k^3 + 6k^2 + 11k + 5 = (k + 1)(k + 2)(k + 3) - 1$

Hence, given limit becomes

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k!}-\frac{1}{(k+3)!}\right)=\frac{5}{3}$$

5. **(a) :** Let $a_n=\prod_{k=0}^{n}\left(1+\frac{2}{x^{2^k}+x^{-2^k}}\right), |x|>1$

$$\Rightarrow a_n=\prod_{k=0}^{n}\frac{(x^{2^k}+1)^2}{x^{2^{k+1}}+1}$$

$$=\frac{(x+1)(x-1)(x+1)(x^2+1)...(x^{2^n}+1)}{(x-1)(x^{2^{n+1}}+1)}$$

$$=\frac{x+1}{x-1}\cdot\frac{x^{2^{n+1}}-1}{x^{2^{n+1}}+1}$$

So, for $|x| > 1$,

$$\therefore\ \lim_{n\to\infty}a_n=\left(\frac{x+1}{x-1}\right)\left(\frac{1-0}{1+0}\right)=\frac{x+1}{x-1}$$

6. **(b) :** Since, $1\le\sqrt[n]{2\sin^2\frac{n^{21}}{n+1}+\cos^2\frac{n^{21}}{n+1}}\le\sqrt[n]{2}$

Hence, required limit (as $n\to\infty$) = 1

7. **(a) :** The given numerator

$$= a(1 + 11 + 111 + ... + \underbrace{111....1}_{n\text{ times}})$$

$$=\frac{a}{9}[(10-1)+(10^2-1)+...+(10^n-1)]$$

$$=\frac{a}{81}\cdot[10(10^n-1)-9n]$$

So, given limit $=\frac{10a}{81}$

8. **(b) :** We have,

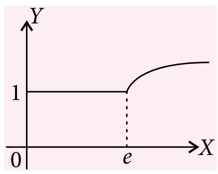
$$\sin^2(\pi\sqrt{n^2+n})=\sin^2[\pi\sqrt{n^2+n}-n\pi+n\pi]$$

$$=\sin^2\left(\frac{\pi}{1+\sqrt{1+\frac{1}{n}}}\right)\to 1\text{ as } n\to\infty$$

9. **(b) :** $f(x)=\lim_{m\to\infty}\frac{\log(e^m+x^m)}{m}$

$$=\lim_{m\to\infty}\frac{m+\log\left(1+\left(\frac{x}{e}\right)^m\right)}{m}$$

$$=\begin{cases}1, \text{if } 0\le x\le e\\ \log x, \text{if } x>e\end{cases}$$



10. **(a) :** Consider the equality,

$$\sum_{k=0}^{2n}(-1)^k\,{}^{2n}C_k\cdot e^{kx}=(e^x-1)^{2n} \qquad ...(i)$$

Now, differentiating (i) n times and putting $x = 0$ gives, $\sum_{k=0}^{2n}(-1)^k\,{}^{2n}C_k\cdot k^n=0$

11. (a) : The given limit can be re-written as

$$L=\lim_{n\to\infty}\left[\frac{f\left(a+\frac{1}{n}\right)-f(a)}{\frac{1}{n}}+2\cdot\frac{f\left(a+\frac{2}{n}\right)-f(a)}{\frac{2}{n}}+.....+k\cdot\frac{f\left(a+\frac{k}{n}\right)-f(a)}{\frac{k}{n}}\right]$$

$= (1 + 2 + 3 + ... + k)f'(a)$

[Limit definition from first principle]

$$=\frac{k(k+1)}{2}f'(a)$$

12. (a) : Rewrite the given limit expression as

$$\lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)^m-1}{\frac{1}{n}}+2\cdot\frac{\left(1+\frac{2}{n}\right)^m-1}{\frac{2}{n}}+...+k\cdot\frac{\left(1+\frac{k}{n}\right)^m-1}{\frac{k}{n}}\right]$$

$$=\frac{k(k+1)}{2}\cdot m$$

13. (a) : Given limit

$$=\lim_{x\to0}\left[\frac{f(x)-f(0)}{x}+\frac{f\left(\frac{x}{2}\right)-f(0)}{x}+...+\frac{f\left(\frac{x}{k}\right)-f(0)}{x}\right]$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{k}\right)f'(0)$$

14. (a) : Simplifying the inequality, we get

$$(f(x)-e^x)(f(x)-e^{x^2})\le 0$$

$$\Rightarrow\ e^{x^2}\le f(x)\le e^x \text{ when } x\in(0,1)$$

$$\Rightarrow\ e^{x}\le f(x)\le e^{x^2} \text{ when } x>1$$

So, $\lim\limits_{x\to1} f(x)=e$

15. (d) : At $x = \pi^+$, $\cot x \to \infty$

So, $2^{\cot x}$, $3^{\cot x}$ and $5^{\cot x} \to \infty$

At $x = \pi^-$, $\cot x \to -\infty$

$\Rightarrow 2^{\cot x}$, $3^{\cot x}$ and $5^{\cot x} \to 0$

$$\text{So, R.H.L}=\lim_{x\to\pi^+}\frac{\left(\frac{2}{5}\right)^{\cot x}+\left(\frac{3}{5}\right)^{\cot x}-5+\frac{2}{5^{\cot x}}}{\left(\frac{4}{5}\right)^{\cot x}+\left(\frac{3}{5}\right)^{\cot x}-1+\frac{1}{5^{\cot x}}}=5$$

and L.H.L = 2

So, limit does not exist at $x = \pi$.

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE Main & Advanced Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE Main & Advanced. In every issue of MT, challenging problems are offered with detailed solution. The reader's comments and suggestions regarding the problems and solutions offered are always welcome.

1. A bag contains 10 white and 3 black balls. Balls are drawn one by one without replacement till all the black balls are drawn. The probability that the procedure of drawing balls will come to an end at the seventh draw is

(a) $\frac{15}{286}$ (b) $\frac{105}{286}$ (c) $\frac{35}{286}$ (d) $\frac{7}{286}$

2. Let $f: R \to R$ be a differentiable function satisfying $f(y)\,f(x-y) = f(x)\ \forall\ x, y \in R$ and $f'(0) = p$, $f'(5) = q$, then $f'(-5)$ is

(a) $\frac{p^2}{q}$ (b) $\frac{p}{q}$ (c) $\frac{q}{p}$ (d) q

3. The sum of all divisors of the least natural number having 12 divisors is

(a) 168 (b) 188
(c) 156 (d) none of these

4. The number of ordered triplets (p, q, r) where $1 \le p, q, r \le 10$ such that $2^p + 3^q + 5^r$ is a multiple of 4, is $(p, q, r \in N)$

(a) 1000 (b) 500 (c) 250 (d) 125

5. A variable triangle is inscribed in a circle of radius R. If the rate of change of a side is R times the rate of change of the opposite angle, then that angle is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

6. The value of $\lim\limits_{n\to\infty}\left(\sqrt[3]{n^2-n^3}+n\right)$ is

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

7. $ABCD$ and $PQRS$ are two variable rectangles such that A, B, C and D lie on PQ, QR, RS and SP respectively and perimeter 'x' of $ABCD$ is constant. If the maximum area of $PQRS$ is 32, then x is equal to

(a) 8 (b) 10 (c) 12 (d) 16

8. In ΔABC, least value of $\frac{e^A}{A}+\frac{e^B}{B}+\frac{e^C}{C}$ is equal to

(a) $\frac{\pi}{3}e^{\pi/3}$ (b) $\frac{\pi}{9}e^{\pi/3}$
(c) $\frac{9}{\pi}e^{\pi/3}$ (d) none of these

9. The probability that the birth days of six different persons will fall in exactly two calendar months is

(a) $\frac{341}{12^5}$ (b) $\frac{66}{12^5}$
(c) $\frac{352}{12^5}$ (d) none of these

10 The number of solutions of the equation $\sin^{-1}\left(\frac{1+x^2}{2x}\right)=\frac{\pi}{2}(\sec(x-1))$ is/are

(a) 1 (b) 2 (c) 3 (d) infinite

SOLUTIONS

1. (a) : Required probability = Probability of getting exactly two black balls and 4 white balls from 1 to 6th draw × probability of getting 3rd black ball in 7th draw.

$$=\frac{{}^{3}C_2\cdot{}^{10}C_4}{{}^{13}C_6}\times\frac{1}{7}=\frac{15}{286}.$$

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2. (a) : Differentiating the functional equation with respect to x, we get $f(y)f'(x-y) = f'(x)$. ...(i)
Put $x = y$ in (i), we get $f'(x) = pf(x)$ ...(ii)

$$\int \frac{f'(x)}{f(x)}dx = \int p\,dx \Rightarrow \log_e f(x) = px + C$$

At $x = 0$, (ii) becomes $f'(0) = pf(0) \Rightarrow f(0) = 1$

$\Rightarrow \log_e f(0) = C \Rightarrow C = 0$

$\therefore f(x) = e^{px} \Rightarrow f'(x) = pe^{px}$

$$\Rightarrow f'(5) = pe^{5p} = q \Rightarrow e^{5p} = \frac{q}{p}$$

and $f'(-5) = pe^{-5p} = \dfrac{p^2}{q}$

3. (a) : $12 = 4 \times 3 = 2 \times 2 \times 3$
The number should be of the form $a^1b^1c^2$ where a, b, c are prime numbers. For number to be least,
$c = 2, b = 3, a = 5$
$\therefore$ Least number $= 2^2\, 3^1\, 5^1 = 60$
Sum of all divisors
$= (2^0 + 2^1 + 2^2)(3^0 + 3^1)(5^0 + 5^1) = (7)(4)(6) = 168.$

4. (b) : $2^p + 3^q + 5^r = 2^p + (4-1)^q + (4+1)^r$
$= 2^p + 4\lambda_1 + (-1)^q + 4\lambda_2 + 1^r$ (λ_1, λ_2 are integers)
If $p = 1$, q should be even and r can be any number. On the other hand if $p \neq 1$, q should be odd and r can be any number.
Total number of ordered triplets
$= 5 \times 10 + 9 \times 5 \times 10 = 500.$

5. (c) : Let side $BC = a$ and A be the opposite angle.

Now, $R = \dfrac{a}{2\sin A} \Rightarrow a = 2R\sin A$

$$\Rightarrow \frac{da}{dt} = 2R\cos A\frac{dA}{dt}$$

$$\Rightarrow R\frac{dA}{dt} = 2R\cos A\frac{dA}{dt} \qquad \left[\because \frac{da}{dt} = R\frac{dA}{dt}\right]$$

$$\therefore \cos A = \frac{1}{2} \Rightarrow A = \frac{\pi}{3}$$

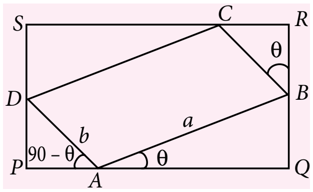
6. (a) : We have, $\lim\limits_{n\to\infty}\left(n\sqrt[3]{\frac{1}{n}-1}+n\right)$

$$= \lim_{n\to\infty} n\left[\left(\frac{1}{n}-1\right)^{1/3} + 1^{1/3}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\left(\frac{1}{n}-1\right)+1}{\left(\frac{1}{n}-1\right)^{2/3} - \left(\frac{1}{n}-1\right)^{1/3} + 1}\right] \qquad \left\{\because a+b = \frac{a^3+b^3}{a^2-ab+b^2}\right\}$$

$$= \lim_{n\to\infty}\left[\frac{1}{\left(\frac{1}{n}-1\right)^{2/3} - \left(\frac{1}{n}-1\right)^{1/3} + 1}\right] = \frac{1}{3}.$$

7. (d) : Here, $2a + 2b = x$



i.e., $a + b = \dfrac{x}{2}$ ...(i)

Area (A') of rectangle $PQRS = PQ \times QR$
$= (PA + AQ) \cdot (QB + BR)$
$= (b\sin\theta + a\cos\theta)(a\sin\theta + b\cos\theta)$

$$= ab + \frac{(a^2+b^2)\sin 2\theta}{2}$$

$$\therefore A' \le ab + \frac{a^2+b^2}{2} \text{ or } A' \le \frac{(a+b)^2}{2}$$

$$A'_{max} = \frac{(a+b)^2}{2} = 32 \Rightarrow \frac{x^2}{8} = 32 \text{ [From (i)]}$$

$\therefore x = 16.$

8. (c) : Using A.M. and G.M. inequality, we get

$$\frac{e^A}{A} + \frac{e^B}{B} + \frac{e^C}{C} \ge 3\left(\frac{e^{A+B+C}}{ABC}\right)^{1/3} \quad ...(i)$$

and $A + B + C \ge 3(ABC)^{1/3}$ ...(ii)

$$\Rightarrow \frac{\pi}{3} \ge (ABC)^{1/3} \Rightarrow \left(\frac{1}{ABC}\right)^{1/3} \ge \frac{3}{\pi}$$

$$\Rightarrow \left(\frac{e^\pi}{ABC}\right)^{1/3} \ge \frac{3}{\pi}e^{\pi/3} \Rightarrow 3\left(\frac{e^\pi}{ABC}\right)^{1/3} \ge \frac{9}{\pi}e^{\pi/3} \quad ...(iii)$$

$\therefore$ From (i) and (iii), we get $\dfrac{e^A}{A} + \dfrac{e^B}{B} + \dfrac{e^C}{C} \ge \dfrac{9}{\pi}e^{\pi/3}.$

9. (a) : Total number of ways in which 6 persons can have their birth days $= 12^6$
Out of 12 months, 2 months can be chosen in ${}^{12}C_2$ ways. Now, birth days of six persons can fall in these two months in 2^6 ways. Out of these 2^6 ways, there are two ways when all six birth days fall in one month. So, there are $(2^6 - 2)$ ways in which six birth days fall in chosen 2 months.

$$\therefore \text{ Required probability} = \frac{{}^{12}C_2(2^6-2)}{12^6} = \frac{341}{12^5}$$

10. (a) : $\sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined for $\left|\dfrac{1+x^2}{2x}\right| \le 1$

$\Rightarrow x = \pm 1$

Out of these two values of x, only $x = 1$ satisfies the given equation.

TARGET JEE

Permutations and Combinations

PROBLEMS

Single Correct Answer Type

1. 7 boys and 7 girls are sit in a row. Then number of ways they can be seated so that girls are separated is

(a) $\binom{14}{7}$ (b) $(7!)^2$ (c) 14! (d) $7! \times 8!$

2. There are 4 copies each of 5 different books. Then number of arrangements by which they can arrange themselves in a shelf is

(a) $\dfrac{20!}{(5!)^4}$ (b) $\dfrac{20!}{(4!)^5}$

(c) $\dfrac{20!}{5!4!}$ (d) None of these

3. If the letters of the word "SCHOOL" are arranged as per dictionary, then rank of the word "SCHOOL" is

(a) 302 (b) 301 (c) 303 (d) 304

4. The number of ways in which 12 different toys can be distributed among three different kids so that youngest kid gets 5, the middle gets 4 and oldest gets 3 is $k \cdot 3^2 \times 2^3$, then k equals

(a) 385 (b) 770

(c) 165 (d) None of these

5. The total number of integers 'n' such that $2 \le n \le 1000$ and H.C.F of 'n' and 36 is one, is equal to

(a) 334 (b) 332 (c) 167 (d) 333

6. Let $A(2, 6)$ and $B(8, 8)$ are two points. Starting from A, line segments of unit length are drawn either right wards or upwards only, untill B is reached, then the number of ways in which A and B connect is 7×2^k. The value of k equals

(a) 4 (b) 3 (c) 2 (d) 1

7. 12 persons are invited for a party. In how many different ways can they and the host be seated at a circular table, if the two particular person are to be seated on either side of the host?

(a) $12! \times 2!$ (b) 11! (c) 9! (d) $2! \times 9!$

8. The number of points (a, b, c) in space whose each coordinate is negative integer such that $a + b + c + 15 = 0$ is

(a) 136 (b) 106 (c) 166 (d) 178

9. The product of all divisors of 1440 is the number α which is divisible by 24^x, then maximum value of x equals

(a) 36 (b) 30 (c) 24 (d) 32

10. If permutations of 4 letters taken from the word 'MATHEMATICS' is A and permutations of 4 letters taken from the word 'EXAMINATION' is B, then $|A - B|$ equals

(a) 0! (b) 3

(c) 2! (d) None of these

More Than One Correct Answer Type

11. Subhash has 10 friends, among them two are married to each other. He wishes to invite 5 of them for a party. If the married couple refused to attend to party without their life partner, then the number of ways in which Subhash can invite five friends is

(a) $2 \times {}^8C_5$ (b) ${}^{10}C_5 - 2 \times {}^8C_4$

(c) ${}^8C_5$ (d) $2 \times {}^8C_3$

12. If n objects are arranged in a row, then the number of ways of selecting three of these objects so that no two of them are adjacent to each other is

(a) ${}^{n+2}C_3$ (b) ${}^{n+1}C_3 + {}^{n+1}C_2$

(c) $\dfrac{(n-2)(n-3)(n-4)}{3!}$ (d) None of these

13. There are 30 students in a class and the prizes to be awarded to the students in such a way that first

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and second goes to mathematics, first and second in Accounts, first goes to Economics and first goes to English. If N denote the number of ways of awarding the prize where $N = (30)^4(K)^2$, then

(a) the value of K is 29 (b) $2700K$ divides N
(c) $200K$ divides N (d) $400K$ divides N

14. Let $A_1, A_2, \ldots, A_{30}$ are 30 sets each set having 5 elements and $B_1, B_2, \ldots, B_n$ are n sets each having three elements such that $\bigcup_{i=1}^{30} A_i = \bigcup_{i=1}^{n} B_i = \lambda$. If each element of λ belongs to exactly ten of the A_i's and exactly 9 of B_i's then the value of n is

(a) 45 (b) ${}^{10}C_8$ (c) ${}^{10}C_2$ (d) 15

15. Let N denote the number of ways in which $3n$ letters can be selected from three sets A, B and C each having $2n$ letters, then

(a) n divides $N - 1$ (b) 3 divides $N - 1$
(c) $(n + 1)$ divides $N - 1$ (d) $3n(n + 1)$ divides $N - 1$

16. ${}^{2n}P_n$ i.e. $P(2n, n)$ is equal to

(a) $n!C(2n, n)$
(b) $(n + 1)(n + 2)\ldots(2n)$
(c) $2^n[(2n - 1)(2n - 3) \ldots 5 \cdot 3 \cdot 1]$
(d) $2 \cdot 6 \cdot 10 \ldots (4n - 2)$

17. Triplets (a, b, c) are chosen from first n natural numbers in such a way that $a \le b < c$, then number of such triplets is

(a) ${}^{n+1}C_3$ (b) ${}^{n}C_2$
(c) $n! - 3$ (d) ${}^{n}C_2 + {}^{n}C_3$

Comprehension Type

Paragraph for Q. No. 18 to 20

Let A be the set of first 18 natural numbers, the number of ways of selecting from A, if

18. Two numbers x and y are such that sum of their cubes is multiple of 3 is

(a) $({}^6C_2)^2 + ({}^6C_1)^2$ (b) ${}^6C_2 + {}^6C_1$
(c) 51 (d) $2({}^6C_2 + {}^6C_1)$

19. Three numbers are such that they form an A.P. is

(a) 72 (b) 144
(c) 216 (d) None of these

20. Three numbers such that, they all are consecutive is

(a) 81 (b) 16 (c) 40 (d) 64

Paragraph for Q. No. 21 to 23

Let p be a prime number and n be a positive integer, then exponent of p in $n!$ is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^k}\right] \text{ where } p^k < n < p^{k+1}$$

and [.] denote the greatest integer function.

21. The number of zeroes at the end of 109! is

(a) 104 (b) 129 (c) 77 (d) 25

22. The last non zero digit in 12! must be equal to

(a) 7 (b) 5 (c) 3 (d) 6

23. The exponent of 11 in ${}^{100}C_{50}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Matrix Match Type

24. Match the following :

	Column-I		Column-II
A.	If n be the number of ways in which 12 different items can be distributed in three groups, then the number $\frac{(4!)^3(3!)^2}{12!}n$ is divisible by	P.	7
B.	If $E_3(19!) = A$, then value of A equals	Q.	9
C.	An n digit number is a positive number with exactly n digits. Nineteen thousand distinct n digit numbers are to be formed using only the three digits 2, 3 and 7. The smallest value of n for which this can be possible is	R.	4
D.	The number of ways in which 9 identical balls can be placed in three identical boxes is $3k$, then the value of k is	S.	6
E.	If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then a value of n is	T.	8

Numerical Answer Type

25. If N is the least natural number which leaves the remainder 2 when divided by 3, 4, 5, 6 and 7, then the number of divisors of N = ...

26. If A and B are respectively combinations and permutations of 4 letters taken from the word "MATHEMATICS", then $|A - B|$ equals

27. If $P(m + n, 2) = 90$ and $P(m - n, 2) = 30$ and $A = m^2 - n^2$, then number of divisors of A = ...

28. Let $S = \{2^0, 2^1, 2^2, ..., 2^{10}\}$. Consider all possible positive differences of elements of S. If M is the sum of all these differences, then number of divisors of the sum of the digits of M is = ...

29. The number of integers greater than 6000 that can be formed by using the digits 3, 5, 6, 7 and 8 if digits are not repeated is _______

30. The letter of the word 'NUMBER' are written out as in a dictionary, the rank of the word NUMBER is _____

SOLUTIONS

1. (d) : Given condition is no two girls sit together.
First arrange 7 boys and they can be seated in 7! ways.

$$\boxed{\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times B_7 \times}$$

Now, there are 8 gaps between 7 boys represented by × and girls can be arranged themselves in $^8P_7$ *i.e.*, 8! ways.
∴ Required number of ways = 7! × 8! ways

2. (b) : Total sets = 5.
Number of identical books in each set = 4
∴ Required number of arrangements $= \dfrac{(5\times4)!}{(4!)^5}$

3. (c) :

Alphabets	S	C	H	O	O	L
Position of alphabets	5	1	2	4	4	3
Number of number less than number of each alphabets	5	0	0	1	1	0
	5!	4!	3!	2!	1!	0!

∴ Required Rank

$$=\left(\frac{5\times5!}{2!}+0+0+\frac{1\times2!}{2!}+\frac{1\times1!}{1!}\right)+1=303$$

4. (a) : Total number of toys = 12
Number of groups are 3 (youngest, middle and oldest) containing 5 toys, 4 toys and 3 toys respectively.
∴ Required number of ways

$$=\frac{12!}{5!4!3!}=385\times2^3\times3^2 \Rightarrow k=385$$

5. (b) : We have $36 = 3^2 \cdot 2^2$ and total numbers from 2 to 1000 = 999
From 2 to 1000, number of multiples of 2 are

$$\frac{1000}{2}=500$$

From 2 to 1000, number of multiples of 3 are

$$\frac{1000}{3}=333$$

From 2 to 1000, number of multiples of 6 are

$$\frac{1000}{6}=166$$

∴ Number of possible values of 'n' are
= 999 – (500 + 333 – 166) = 332

6. (c) : ∵ $A(x, y) = (2, 6)$, $B(x, y) = (8, 8)$
∵ Difference of x-coordinates = 8 – 2 = 6 and that of y co-ordinates = 8 – 6 = 2
∴ Exactly 6 steps rightwards and 2 steps upward are required.
Let rightward step is denoted by R and upward step by U.
Now, we need to arrange the letters $R\,R\,R\,R\,R\,R\,U\,U$
∴ Number of arrangements by which A and B connect to each other $=\dfrac{(6+2)!}{6!2!}=28=7\times2^k \Rightarrow k=2.$

7. (d) : Total person for party = 12 + 1 (host) = 13. Now, the host and two particular persons to be consider as one unit therefore there remains 12 – 3 + 1 = 10 persons and they can be arrange themselves in 9! ways and two persons on either side of the host can arrange themselves in 2! ways.
∴ Required arrangements = 2! × 9!

8. (a) : Let $a = -x$, $b = -y$, $c = -z$, where x, y, z are non negative integers.
Now, $x + y + z = 15$
A required number of points (a, b, c) is the number of non negative integer solution of $x + y + z = 15$

$$= {}^{17}C_2 = \frac{17\times16}{2\times1}=136$$

9. (b) : ∵ $1440 = 32 \times 9 \times 5 = 2^5 3^2 5^1$
∴ Number of divisors of 1440 are (5 + 1) (2 + 1) (1 + 1) = 36
∴ Product of divisors = (1, 2, 3, 4 ,....., 360, 480, 720, 1440) which are in increasing order and now clubbed in to 18 pairs as (1,1440), (2,720), (3,480), (4,360), (5,288) .... etc (18 pairs)
$= (1440)^{18} = 2^{90} \cdot 3^{36} \cdot 5^{18} = (2^3)^{30} \cdot 3^{30} \cdot 3^6 \cdot 5^{18}$
∴ $(1440)^{18} = 24^{30} \cdot 3^6 \cdot 5^{18}$, which is divisible by 24^x
∴ Maximum value of $x = 30$

10. (d) : In word 'MATHEMATICS' we have $M = 2$, $A = 2$, $T = 2$ and remaining 5 letters are distinct, and in the word 'EXAMINATION' there are $A = 2$, $N = 2$, $I = 2$ and remaining 5 letters are distinct. Therefore the number of permutations of 4 letters taken from either of two word is same
∴ $A = B \Rightarrow |A - B| = 0.$

11. (a, b, d) : Total friends = 10.
Married = 2 and bachelor = 10 – 2 = 8

Now, number of ways of inviting five friends including partners or couple = $^2C_2 \times {}^8C_3$.
Also, number of ways of inviting the friends excluding the partners = $^8C_5$
$\therefore$ Total ways of inviting his friends = $^8C_3 + {}^8C_5$
$= 2 \cdot {}^8C_3 = 2 \cdot {}^8C_5$

$$\text{Again, } {}^{10}C_5 - 2 \times {}^8C_4 = \frac{10}{5}({}^9C_4) - 2 \times {}^8C_4$$

$= 2({}^9C_4 - {}^8C_4) = 2 \times ({}^8C_3 + {}^8C_4 - {}^8C_4) = 2 \cdot {}^8C_3$

12. (a, b) : Let l be the number of objects to the left of the first object chosen, l_2 be the number of objects between first and second, l_3 be the number of objects between the second and third object and l_4 be the number of objects right to third object.
$\therefore$ $l_1, l_4 \geq 0$ and $l_2, l_3 \geq 1$
Also, $l_1 + l_2 + l_3 + l_4 = n - 3$ ...(i)
$\because$ $(1 + l_1) + l_2 + l_3 + (1 + l_4) = n - 1$ ...(ii)
Here $r = 4$,
$\therefore$ Required number of ways = number of solution of (ii)
$= {}^{(n-1)+(4-1)}C_{4-1}$
$= {}^{n+2}C_3 = {}^{n+1}C_3 + {}^{n+1}C_2$

13. (a, b, c, d) : The first and second prizes which goes to Mathematics as well as Accounts can be awarded in $^{30}P_2 \times {}^{30}P_2$ ways = $30^2 \times 29^2$ ways.
Again, first prize goes to Economics as well as English can be awarded by $^{30}P_1 \times {}^{30}P_1 = (30)^2$ ways
$\therefore$ $N = (30)^2(29)^2(30)^2 = (30)^4(29)^2 = (30)^4(K)^2$
$\Rightarrow$ $K = 29$
Again $N = (30)^4 \cdot (29)^2 = 3^4 \cdot 5^4 \cdot 2^4(29)^2 = 2700(300)(29)^2 = 2700K\,(300 \times 29)$
Similarly $N = (30)^4(29)^2 = (2 \times 3 \times 5)^4(29)^2$
$= 2^3 5^2 29(2^1 \cdot 5^2 \cdot 3^4\, 29) = 200K(2^1 \cdot 5^2 \cdot 3^4 \cdot 29)$ which is divisible by $200K = 400K(5^2 \cdot 3^4 \cdot 29)$ which is divisible by $400K$.

14. (a, b, c) : Since each A_i has 5 element, we have

$$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150 \quad \text{...(i)}$$

Suppose λ has m distinct elements.
Since each element of 'λ' belongs to exactly 10 of A_i's
$\therefore$ we also have

$$\sum_{i=1}^{30} n(A_i) = 10m \quad \text{...(ii)}$$

Now on comparing (i) and (ii), we have $m = 15$
Since B_i has 3 elements and each element of 'λ' belongs to exactly 9 of B_i's

$$\therefore \quad \sum_{i=1}^{n} n(B_i) = 3n \text{ and } \sum_{i=1}^{n} n(B_i) = 9m$$

$\Rightarrow$ $3n = 9m \Rightarrow n = 3m \Rightarrow n = 3 \times 15 = 45 = {}^{10}C_8 = {}^{10}C_2$

15. (a, b, c, d) : N = coefficient of x^{3n} in $(1 + x + x^2 + ... + x^{2n})(1 + x + x^2 + ... + x^{2n})(1 + x + x^2 + ... + x^{2n})$
= coefficient of x^{3n} in $(1 + x + x^2 + ... + x^{2n})^3$
= coefficient of x^{3n} in $(1 - x^{2n+1})^3(1 - x)^{-3}$
= coefficient of x^{3n} in $(1 - 3x^{2n+1}$ + higher terms of $x^{3n})(1 + {}^3C_1x^1 + {}^4C_2x^2 + ... + {}^{n+1}C_{n-1}x^{n-1} + ... + {}^{3n+2}C_{3n}x^{3n}) = {}^{3n+2}C_{3n} - 3.\,{}^{n+1}C_{n-1}$
$\Rightarrow$ $N = {}^{3n+2}C_2 - 3.{}^{n+1}C_2$

$$= \frac{1}{2}\{(9n^2 + 9n + 2) - (3n^2 + 3n)\} = \frac{1}{2}[6n^2 + 6n + 2]$$

$\therefore$ $N - 1 = 3n^2 + 3n = 3n(n + 1)$ and $N - 1$ has factors, 3, n, $n + 1$, $3n(n + 1)$

16. (a, b, c, d) : $^{2n}P_n = \dfrac{2n!}{n!} = n!\dfrac{(2n)!}{n!n!} = n!C(2n, n)$

$$\text{Again, } {}^{2n}P_n = \frac{2n!}{n!}$$

$$= \frac{(2n)(2n-1)(2n-2)(2n-3)....5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n!}$$

$$= \frac{[(2n)(2n-2)(2n-4)....4 \cdot 2]}{n!}[(2n-1)(2n-3)...5 \cdot 3 \cdot 1]$$

$$= 2^n \frac{[(n)(n-1)(n-2)...2 \cdot 1]}{n!}[(2n-1)(2n-3)....5 \cdot 3 \cdot 1]$$

$= 2^n[(2n - 1)(2n - 3)(2n - 5)\ ....5 \cdot 3 \cdot 1]$
$= [(4n - 2)(4n - 6)(4n - 10)\ .....\ 10 \cdot 6 \cdot 2]$
$= 2 \cdot 6 \cdot 10\ ......\ (4n - 6) \cdot (4n - 2)$

$$\text{Again } {}^{2n}P_n = \frac{(2n)!}{n!}$$

$$= \frac{(2n)(2n-1)(2n-2).....(n+1)n!}{n!} = (n+1)(n+2)\ .....\ (2n)$$

17. (a, d) : If $a < b < c$, then number of selections = $^nC_3$
If $a = b < c$, then number of such selection = $^nC_2$
$\therefore$ Required numbers = $^nC_2 + {}^nC_3 = {}^{n+1}C_3$

18. (c) : Write down the numbers in rows as

$$\left.\begin{matrix} 1 & 4 & 7 & 10 & 13 & 16 \\ 2 & 5 & 8 & 11 & 14 & 17 \\ 3 & 6 & 9 & 12 & 15 & 18 \end{matrix}\right\} \text{.....(each row having 6 numbers)}$$

Now, sum of their cubes *i.e.*, $x^3 + y^3$ is divisible by 3. If $x + y$ is multiple of 3, then the numbers x and y are drawn either both from third row or one from first row and other from second row.
$\therefore$ Required ways = $^6C_2 + {}^6C_1 \cdot {}^6C_1 = 15 + 36 = 51$

19. (a) : Let x, y, z are in A.P $\Rightarrow x + z = 2y$(even number).
As $x + z$ is even there exist two cases, either both x and z are even or both are odd. Now, there are 9 odd numbers and 9 even numbers in the set.
$\therefore$ Required ways = $^9C_2 + {}^9C_2 = 2({}^9C_2) = 72$

20. (b) : Given set $A = \{1, 2, 3, ... 16, 17, 18\}$

When three number are consecutive, the number of such sets are (1 2 3), (2 3 4), (3 4 5), (4 5 6).... (15 16 17), (16 17 18). Hence there are 16 ways.

$\therefore$ Required ways = 16

21. (d) : $E_2(109!) = \left[\frac{109}{2}\right] + \left[\frac{109}{4}\right] + \left[\frac{109}{8}\right] + \left[\frac{109}{16}\right] + \left[\frac{109}{32}\right] + \left[\frac{109}{64}\right] + 0$

$= 54 + 27 + 13 + 6 + 3 + 1 = 104$

and $E_5(109)! = \left[\frac{109}{5}\right] + \left[\frac{109}{25}\right] + 0 = 21 + 4 = 25$

$\therefore$ Number of zero's at the end of 109! = minimum value of $\{104, 25\} = 25$

22. (d) : We given $x! = 12!$ thus number of prime up to 12 are 2, 3, 5, 7, 11.

$\therefore$ $E_2(12!) = \left[\frac{12}{2}\right] + \left[\frac{12}{4}\right] + \left[\frac{12}{8}\right] = 6 + 3 + 1 = 10$

$E_3(12!) = \left[\frac{12}{3}\right] + \left[\frac{12}{9}\right] = 4 + 1 = 5$

$E_5(12!) = 2$, $E_7(12!) = E_{11}(12!) = 1$

$\therefore$ $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1 = 2^2 \times 5^2[2^8 \cdot 3^5 \cdot 7 \cdot 11]$

Now, 2^8 ends with 6 and 3^5 ends with 3

$\therefore$ Last non zero digit in 12! = last digit in the product $[6 \cdot 3 \cdot 7 \cdot 11]$ and the last digit is 6.

$\therefore$ Last non zero digit in 12! = 6.

23. (b) : We have, ${}^{100}C_{50} = \frac{100!}{50!50!}$

$\therefore$ $E_{11}(100!) = \left[\frac{100}{11}\right] + \left[\frac{100}{11^2}\right] = 9$

$\therefore$ $E_{11}(50!) = \left[\frac{50}{11}\right] + \left[\frac{100}{11^2}\right] = 4$

$\therefore$ Exponent of 11 in ${}^{100}C_{50} = 9 - (4 \times 2) = 1$

24. A $\to$ S, B $\to$ T, C $\to$ Q, D $\to$ R, E $\to$ P

Total items = 12(All different)

Number of groups = 3 in which item to be distributed so each group to be assign 4 items.

A. Now out of 12 items each group assigns 4 items so total number of ways

$\therefore$ $n = \frac{12!}{(4!)^3 \cdot 3!}$

$\Rightarrow$ $\frac{(4!)^3 3! n}{12!} = 1 \Rightarrow \frac{(4!)^3 (3!)^2 \cdot n}{12!} = 3! = 6$

B. $E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ... + \left[\frac{n}{p^k}\right]$,

$p^k < n < p^{k+1}$, $[x]$ denotes integer part of x.

$\therefore$ $E_3(19!) = \left[\frac{19}{3}\right] + \left[\frac{19}{9}\right] + 0 = 6 + 2 = 8$

C. Given number is used to form nineteen thousand distinct n digit numbers using 2, 3 and 7.

Now, each place can be filled by 3 ways by using 2, 3, 7

$\therefore$ n digit number can be formed by 3^n ways

$\therefore$ $3^n \geq 19000$

We know that $3^8 = 9^4 = 6561$ and $3^9 = 19683 > 19000$

$\therefore$ Smallest value of $n = 9$

D. Required ways = ${}^3C_1 \times {}^2C_1 \times {}^1C_1 \times 2 = 12$

$\Rightarrow$ $3k = 12 \Rightarrow k = 4$

E. As ${}^nC_4$, ${}^nC_5$, ${}^nC_6$ are in A.P.

$\Rightarrow$ $2 \cdot {}^nC_5 = {}^nC_4 + {}^nC_6$

$\Rightarrow$ $\frac{2 \cdot n!}{5!(n-5)!} = \frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!}$

$\Rightarrow$ $n^2 - 21n + 98 = 0$

$\Rightarrow$ $(n - 7)(n - 14) = 0 \Rightarrow n = 7$ or $n = 14$

25. (4) : We first need to find N, which is [LCM of (3, 4, 5, 6, 7) + 2] = 422

$\therefore$ $N = 422 = 2^1 \times (211)^1$

$\therefore$ Number of divisors of $N = 2 \times 2 = 4$

26. (2318) : In the word "MATHEMATICS" there are A = 2 = M = T and 5 alphabets H, E, I, C, S are distinct (each one time)

$\therefore$ Number of combination = coefficient of x^4 in $(1 + x + x^2)^3 \cdot (1 + x)^5$ = Coefficient of x^4 in $\{(1 + x)^3 + 3(1 + x)x^2(1 + x + x^2) + x^6\}(1 + x)^5$

= coefficient of x^4 in $\{(1 + x)^3 + 3x^2(1 + x)^2 + 3x^4(1 + x) + x^6\}(1 + x)^5$

FUN FACT

There are just four numbers (after 1) which are the sums of the cubes of their digits :

$153 = 1^3 + 5^3 + 3^3$

$370 = 3^3 + 7^3 + 0^3$

$371 = 3^3 + 7^3 + 1^3$

$407 = 4^3 + 0^3 + 7^3$

= coefficient of x^4 in

$(1 + x)^8 + 3x^2(1 + x)^7 + 3x^4(1 + x)^6 + x^6(1 + x)^5$

$= {}^8C_4 + 3 \cdot {}^7C_2 + 3(1) + 0 = 70 + 63 + 3 = 136 = A$

Again, the number of permutation

$$= \text{coeff. of } x^4 \text{ in } (4!)\left\{\left(1+x+\frac{x^2}{2!}\right)^3\left(1+\frac{x}{1!}\right)^5\right\}$$

= coeficient of x^4 in

$$4!\left\{(1+x)^3+3\frac{x^2}{2}(1+x)\left(1+x+\frac{x^2}{2!}\right)+\frac{x^6}{2^3}\right\}(1+x)^5$$

= Coefficient of x^4 in

$$4!\left\{(1+x)^3+\frac{3x^2}{2}(1+x)^2+\frac{3}{4}x^4(1+x)+\frac{x^6}{8}\right\}(1+x)^5$$

= coefficient of x^4 in

$$4!\left\{(1+x)^8+\frac{3}{2}(x^2)(1+x)^7+\frac{3}{4}x^4(1+x)^6\right\}$$

$$\left[\because \text{ there is no } x^4 \text{ in } \frac{x^6}{8}(1+x)\right].$$

$$=(4!)\left[{}^8C_4+\frac{3}{2}{}^7C_2+\frac{3}{4}\right]=2454=B$$

$\therefore \quad |A - B| = 2318$

27. (12) : $P(m + n, 2) = 90$

$\Rightarrow \quad (m + n)(m + n - 1) = 90$

$\Rightarrow \quad (m + n)^2 - (m + n) - 90 = 0$

$\Rightarrow \quad (m + n - 10)(m + n + 9) = 0$

$\Rightarrow \quad m + n = 10 \; [\because m + n \neq -9]$

Again, $P(m - n, 2) = 30 \Rightarrow (m - n)(m - n - 1) = 30$

$= (m - n)^2 - (m - n) - 30 = 0$

$\Rightarrow \quad (m - n - 6)(m - n + 5) = 0$

$\Rightarrow \quad m - n = 6 \; [\because m - n \neq -5]$

Now, $A = m^2 - n^2 = 60 = 2^2 \times 3^1 \times 5^1$

$\therefore$ Number of divisors of $A = 3 \times 2 \times 2 = 12$

28. (4) : $S = \{2^0, 2^1, 2^2, \ldots\ldots 2^{10}\}$

$$\therefore \; M=\sum_{s=1}^{10}\sum_{r=0}^{s-1}(2^s-2^r)$$

$$=\sum_{s=1}^{10}(2^s-2^0)+(2^s-2^1)+\ldots\ldots+(2^s-2^{s-1})$$

$$=\sum_{s=1}^{10}\underbrace{(2^s+2^s+\ldots.+2^s)}_{s\text{ times}}-(2^0+2^1+\ldots.+2^{s-1})$$

$$=\sum_{s=1}^{10}s\cdot 2^s-(1+2^1+2^2+\ldots\ldots+2^{s-1})$$

$$=\sum_{s=1}^{10}s\cdot 2^s-(2^s-1) \quad \text{(using sum of G.P. series)}$$

$$=\sum_{s=1}^{10}[2^s(s-1)+1]$$

$= [2^1(1 - 1) + 1] + [2^2(2 - 1) + 1] + [2^3(3 - 1) + 1] + \ldots. + [2^{10}(10 - 1) + 1]$

$= 2^2(2^0 \cdot 1 + 2^1 \cdot 2 + 2^2 \cdot 3 + \ldots. + 2^8 \cdot 9) + 10$

$= 2^2[8 \cdot 2^9 + 1] + 10 = 4(8 \cdot 2^9 + 1) + 10$

(Using sum of A.G. P of n terms, $n = 9$ and $r = 2$)

$$S_n=\frac{a}{1-r}+\frac{ar(1-r^{n-1})}{(1-r)^2}-\frac{[a+(n-1)d]r^n}{1-r}$$

$= 8 \cdot 2^{11} + 14 = 2^{14} + 14 = 16384 + 14 = 16398$

$\therefore$ Sum of the digit of $M = 1 + 6 + 3 + 9 + 8 = 27 = 3^3$

$\therefore$ Number of divisors of $M = 4$

29. (192) : Given digits are 3, 5, 6, 7 and 8

We need 4 digit numbers greater than 6000 and 5 digit numbers and will add them to get the result.

Now, 5 digit numbers $= {}^5P_5 = 5! = 120$

Now, for 4 digit numbers each greater than 6000, left extreme place can be filled in ${}^3C_1$ ways as this place has choices with 6, 7, 8 digits. Now the remaining three places can be filled by choices 4, 3 and 2 ways.

$\therefore$ Required such numbers $= {}^3C_1 \times 4 \times 3 \times 2 = 72$

$\therefore$ Required numbers greater than 6000

$= 120 + 72 = 192$

30. (469) : Write down the letters of the word as below:

Step-I : Write down position of letters according to alphabets.

Step-II : Write down number of small numbers to the right starting from left (i.e. smaller than 4 are 3, 1, 2 i.e. 3 numbers below R_1.

Step-III : Write 0!, 1!, ... so on, from right below R_2.

Step-IV : Multiply R_2 and R_3 columnwise and add them we get all number before the word 'NUMBER'.

$3 \times 5! + 4 \times 4! + 2 \times 3! + 0 \times 2! + 0 \times 1! + 0 \times 0!$

$= 468$

Step-V : $\therefore$ Required rank is $468 + 1 = 469$

N	U	M	B	E	R	
4	6	3	1	2	5	R_1
3	4	2	0	0	0	R_2
5!	4!	3!	2!	1!	0!	R_3

Note: Use this technique when all alphabets are distinct in the given word.

MOCK TEST PAPER 2019

JEE MAIN

Time : 1 hr 15 min. **Series - 3**

The entire syllabus of Mathematics of JEE MAIN is being divided in to eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:

	Topic	Syllabus In Details
UNIT NO. 3	Permutations & Combinations	Fundamental principle of counting, permutation as an arrangement and combination as selection, meaning of $P(n, r)$ and $C(n, r)$, simple applications.
	Trigonometry	General solution and Properties of Triangle.
	Co-ordinate Geometry-2D	Circles: Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equations of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent.

1. The number 24! is divisible by

(a) 6^{24} (b) 24^6 (c) 12^{12} (d) 48^5

2. The value of $^{35}C_8 + \sum_{r=1}^{7} {^{42-r}C_7} + \sum_{s=1}^{5} {^{47-s}C_{40-s}}$ is

(a) $^{47}C_7$ (b) $^{46}C_8$

(c) $^{47}C_6$ (d) $^{47}C_8$

3. Eighteen guests have to be seated, half on each side of a long table. Four particular guest desires to sit on one particular side and three others on the other side. The number of ways in which the seating arrangement can be made, is

(a) $9! \times 9!$ (b) $^{11}C_5 \times 9! \times 9!$

(c) $\frac{11!}{5!} \times 9! \times 9!$ (d) $^{11}C_5$

4. The number of arrangements that can be made with the letters of the word 'MATHEMATICS' in which all the vowels come together, is

(a) $\frac{8! \times 4!}{2!\,2!}$ (b) $\frac{8! \times 4!}{2!\,2!\,2!}$

(c) $\frac{8!}{2!\,2!\,2!}$ (d) $\frac{8!}{4!\,2!\,2!}$

5. The letters of the word "RANDOM" are written in all possible orders and these words are written out as in a dictionary, then the rank of the word "RANDOM" is

(a) 614 (b) 615 (c) 613 (d) 616

6. The sum of all five digit numbers that can be formed using the digits 1, 2, 3, 4, 5, when repetition of digits is not allowed, is

(a) 366000 (b) 660000

(c) 360000 (d) 3999960.

7. The number of triangles whose vertices are at the vertices of an octagon but none of whose sides happen to come from the sides of the octagon is

(a) 24 (b) 52 (c) 48 (d) 16.

8. The number of odd proper divisors of 5040 is

(a) 12 (b) 10

(c) 11 (d) none of these.

9. There are five balls of different colours and five boxes of colours same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that exactly one ball goes to a box of its own colour, is

(a) 9 (b) 24 (c) 45 (d) 120.

By : Sankar Ghosh, S.G.M.C, Mob. 09831244397.

10. The number of ways to give 16 different things to three persons A, B, C so that B gets 1 more than A and C gets 2 more than B, is

(a) $\dfrac{16!}{4!\,5!\,7!}$ (b) 4! 5! 7!

(c) $\dfrac{16!}{3!\,5!\,8!}$ (d) none of these.

11. The values of x satisfying

$$\cos 2x = (\sqrt{2}+1)\left(\cos x - \frac{1}{\sqrt{2}}\right),\ \cos x \neq \frac{1}{2} \text{ is}$$

(a) $\left\{2n\pi \pm \dfrac{\pi}{3} : n \in Z\right\}$ (b) $\left\{2n\pi \pm \dfrac{\pi}{6} : n \in Z\right\}$

(c) $\left\{2n\pi \pm \dfrac{\pi}{2} : n \in Z\right\}$ (d) $\left\{2n\pi \pm \dfrac{\pi}{4} : n \in Z\right\}$

12. If $\begin{vmatrix} \cos(A+B) & -\sin(A+B) & \cos 2B \\ \sin A & \cos A & \sin B \\ -\cos A & \sin A & \cos B \end{vmatrix} = 0,$

then $B =$

(a) $(2n+1)\dfrac{\pi}{2}, n \in Z$ (b) $n\pi, n \in Z$

(c) $(2n+1)\pi, n \in Z$ (d) $2n\pi, n \in Z$

13. The smallest positive values of x and y, satisfying $x - y = \dfrac{\pi}{4}$ and $\cot x + \cot y = 2$, are

(a) $x = \dfrac{\pi}{6}, y = \dfrac{5\pi}{12}$ (b) $x = \dfrac{5\pi}{12}, y = \dfrac{\pi}{6}$

(c) $x = \dfrac{\pi}{3}, y = \dfrac{7\pi}{12}$ (d) none of these.

14. Number of values of x satisfying the equation $|\sin x| = \sin x + 3$ in $[0, 2\pi]$ are

(a) 0 (b) 1

(c) 2 (d) more than one

15. The number of values of θ in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ such that $\theta \neq \dfrac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is

(a) 4 (b) 5 (c) 7 (d) 3

16. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$ square units. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to

(a) 2 (b) 4 (c) 3 (d) 6

17. In a ΔABC, if $\sin A \cos B = \dfrac{1}{4}$ and $3\tan A = \tan B$, then the triangle is

(a) right angled at A (b) right angled at B

(c) right angled at C (d) not right angled.

18. In ΔABC, AD and BE are the medians drawn through the angular points A and B respectively. $\angle DAB = 2\angle ABE = 36°$ and $AD = 6$ units, then circumradius of the triangle is equal to

(a) $(3-\sqrt{5})\operatorname{cosec}C$ (b) $(3+\sqrt{5})\operatorname{cosec}C$

(c) $2(3-\sqrt{5})\operatorname{cosec}C$ (d) $2(3+\sqrt{5})\operatorname{cosec}C$

19. If the inradius of a circle inscribed in an isosceles triangle whose one angle is $\dfrac{2\pi}{3}$ is $\sqrt{3}$, then the area of the triangle in square units is

(a) $7+12\sqrt{3}$ (b) $12-7\sqrt{3}$

(c) $12+7\sqrt{3}$ (d) 4π

20. Length of two sides of a triangle are given by the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$. The angle between the sides is $\dfrac{\pi}{3}$. The perimeter of the triangle is

(a) $6+\sqrt{3}$ (b) $2\sqrt{3}+\sqrt{6}$

(c) $2\sqrt{3}+\sqrt{10}$ (d) none of these.

21. C_1 and C_2 are two circles touching each other and the coordinate axes. If C_1 is smaller than C_2 and its radius is 2 units, then radius of C_2 is

(a) $6+4\sqrt{2}$ (b) $2+2\sqrt{2}$

(c) $3+2\sqrt{2}$ (d) none of these.

22. The image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ in the mirror of $4x + 7y + 13 = 0$, is

(a) $(x+16)^2 + (y+2)^2 = 5^2$

(b) $(x-16)^2 + (y-2)^2 = 5^2$

(c) $(x+16)^2 + (y-2)^2 = 5^2$

(d) $(x-16)^2 + (y+2)^2 = 5^2$

23. If (2, 4) is a point interior to the circle $x^2 + y^2 - 6x - 10y + \lambda = 0$ and the circle does not cut the axes at any point , then

(a) $\lambda \in (25, 32)$ (b) $\lambda \in (9, 32)$

(c) $\lambda \in (32, \infty)$ (d) none of these.

24. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1 ?

(a) $x + y = 0, x - 7y = 0$ (b) $x - y = 0, x + 7y = 0$

(c) $7x + y = 0$ (d) $x - 7y = 0$

25. The line $4x - 3y = -12$ is tangent at the point $(-3, 0)$ and the line $3x + 4y = 16$ is tangent at the point $(4, 1)$ to a circle. The equation of the circle is

(a) $x^2 + y^2 - 2x + 6y - 15 = 0$
(b) $x^2 + y^2 - 2x + 6y - 20 = 0$
(c) $x^2 + y^2 + 2x + 6y - 15 = 0$
(d) $x^2 + y^2 - 2x - 6y - 15 = 0$

26. The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. The equation of the locus of the point P ,is

(a) $x^2 + y^2 + 4x - 6y + 4 = 0$
(b) $x^2 + y^2 + 4x - 6y - 9 = 0$
(c) $x^2 + y^2 + 4x - 6y - 4 = 0$
(d) $x^2 + y^2 + 4x - 6y + 9 = 0$

27. The angle between the tangents from (α, β) to the circle $x^2 + y^2 = a^2$ is

(a) $\tan^{-1}\left(\frac{a}{\sqrt{s_1}}\right)$ (b) $2\tan^{-1}\left(\frac{a}{\sqrt{s_1}}\right)$

(c) $2\tan^{-1}\left(\frac{\sqrt{s_1}}{a}\right)$ (d) none of these.

Where $s_1 = \alpha^2 + \beta^2 - a^2$.

28. From a point $A(1, 1)$ on the circle $x^2 + y^2 - 4x - 4y + 6 = 0$, two chords AB and AC each of length 2 units are drawn. The equation of chord BC is

(a) $4x + 3y - 12 = 0$ (b) $x + y = 4$
(c) $3x + 4y = 4$ (d) $x + y = 6$

29. Two concentric circles of which smallest is $x^2 + y^2 = 4$, have the difference in radii as d. If line $y = x + 1$ cuts the circles in real points, then d lies in the interval

(a) $\left(-\infty, -2-\frac{1}{\sqrt{2}}\right) \cup \left(-2+\frac{1}{\sqrt{2}}, \infty\right)$

(b) $\left(-2+\frac{1}{\sqrt{2}}, 2+\frac{1}{\sqrt{2}}\right)$

(c) $\left(-\infty, 1-\frac{1}{\sqrt{2}}\right) \cup \left(1+\frac{1}{\sqrt{2}}, \infty\right)$

(d) $\left(1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$

30. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are diameters of a circle of area 49π square units, the equation of the circle is

(a) $x^2 + y^2 + 2x - 2y - 62 = 0$
(b) $x^2 + y^2 - 2x + 2y - 62 = 0$
(c) $x^2 + y^2 - 2x + 2y - 47 = 0$
(d) $x^2 + y^2 + 2x - 2y - 47 = 0$

SOLUTIONS

1. (b) : Exponents of 2 in 24! is

$$E_2(24!) = \left[\frac{24}{2}\right] + \left[\frac{24}{2^2}\right] + \left[\frac{24}{2^3}\right] + \left[\frac{24}{2^4}\right]$$
$$= 12 + 6 + 3 + 1 = 22$$

Similarly, $E_3(24!) = \left[\frac{24}{3}\right] + \left[\frac{24}{3^2}\right] = 8 + 2 = 10$

$\therefore$ $24! = 2^{22} \times 3^{10} = (2^3)^7 \times 3^{10} \times 2$
$= (2^3 \times 3)^7 \times 3^3 \times 2 = (24)^7 \times 3^3 \times 2$

Clearly, 24! is divisible by $(24)^6$.

2. (d) : We have, ${}^{35}C_8 + \sum_{r=1}^{7} {}^{42-r}C_7 + \sum_{s=1}^{5} {}^{47-s}C_{40-s}$

$= {}^{35}C_8 + {}^{35}C_7 + {}^{36}C_7 + {}^{37}C_7 + ... + {}^{41}C_7 + {}^{42}C_{35} + {}^{43}C_{36} + ... + {}^{46}C_{39}$
$= ({}^{35}C_8 + {}^{35}C_7) + {}^{36}C_7 + {}^{37}C_7 + ... + {}^{41}C_7 + {}^{42}C_{35} + {}^{43}C_{36} + ... + {}^{46}C_{39}$
$= ({}^{36}C_8 + {}^{36}C_7) + ... + {}^{41}C_7 + {}^{42}C_7 + {}^{43}C_7 + ...+ {}^{46}C_7$
$[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \text{ and } {}^nC_r = {}^nC_{n-r}]$
$= ({}^{41}C_8 + {}^{41}C_7) + {}^{42}C_7 + {}^{43}C_7 + ... {}^{46}C_7$
$= ({}^{42}C_8 + {}^{42}C_7) + ... + {}^{46}C_7$
$= ....$
$= {}^{46}C_8 + {}^{46}C_7 = {}^{47}C_8$

3. (b) : Given that 4 particular guests will sit on a particular side A (say) and three other on the other side B (say). Therefore, we are to arrange 11 guests so that 5 guests will sit on side A and remaining 6 will sit on side B. This can be done in the following ways—

${}^{11}C_5 \times {}^6C_6$

Now 9 guests on each side can arrange among themselves in 9! ways.

Therefore, total number of arrangements
$= {}^{11}C_5 \times {}^6C_6 \times 9! \times 9!$

4. (b) : In the given word we find there are 4 vowels-A, E, A, I. Now let us consider these 4 vowels (A, E, A, I) as a single letter. In consequence of that the given word "MATHEMATICS" becomes (A, E, A, I) "MATHEMATICS".

Hence, the total number of arrangements in which vowels are always together $= \frac{8!}{2! \times 2!} \times \frac{4!}{2!}$.

5. (a) : In a dictionary the words at each stage are arranged in alphabetical order. So here we consider the words beginning with A, D, M, N, O and R in order. Now keeping A at the first place, the remaining 5 letters can be arranged in 5! ways. Similarly, D, M, N, O will occur in the first place the same number of times *i.e.*, 5!

$\therefore$ Number of words starting with A = 5! = 120
Number of words starting with D = 5! = 120

Number of words starting with M = 5! = 120
Number of words starting with N = 5! = 120
Number of words starting with O = 5! = 120
Number of words beginning with R is 5!, but one of these words is the word RANDOM. So, we first find the number of words beginning with RAD and RAM
Number of words starting with RAD = 3! = 6
and number of words starting with RAM = 3! = 6.
Now, the words beginning with RAN must follow. The first word beginning with RAN is the word RANDMO and the next word is RANDOM.
$\therefore$ Rank of RANDOM = 5 × 120 + 2 × 6 + 2 = 614

6. (d) : We know that the sum of all n-digit numbers formed by using n digits from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 is = (Sum of the digits) $(n-1)!\left(\frac{10^n-1}{10-1}\right)$

$\therefore$ Required sum = $4!\,(1+2+3+4+5)\times\left(\frac{10^5-1}{10-1}\right)$

$= 360\left(\frac{100000-1}{9}\right) = 3999960$

7. (d) : The number of triangles = Total number of triangles – (number of triangles having one side common with the octagon) – (number of triangles having two sides in common)

$= {}^8C_3 - ({}^8C_1 \times {}^4C_1) - 8 = 16$

8. (c) : Here, $5040 = 2^4 \times 3^2 \times 5 \times 7$

$\therefore$ Number of odd proper divisors
$= (2+1)(1+1)(1+1) - 1 = 11$

9. (c) : Let $b_1, b_2, ..., b_5$ be five balls and $B_1, B_2, ...$ B_5 be five boxes. Then, the required number of ways

$$= \left[\sum_{r=1}^{5}(r^{\text{th}} \text{ ball is placed in } r^{\text{th}} \text{box})\right]$$
$$\times \text{ (Remaining balls are placed in wrong boxes)}$$

$$= \sum_{r=1}^{5} 1\times 4!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right\}$$

$$= \sum_{r=1}^{5}(12-4+1) = 45$$

10. (a) : Let A gets x things, then B gets $x + 1$ and C gets $x + 3$ number of things.

$\therefore$ $x + x + 1 + x + 3 = 16 \Rightarrow x + 4$

Thus, we have to distribute 16 things to A, B and C in such a way that A gets 4 things, B gets 5 things and C gets 7 things.

Required number of ways = ${}^{16}C_4 \times {}^{12}C_5 \times {}^7C_7 = \frac{16!}{4!5!7!}$

11. (d) : We have, $\cos 2x = (\sqrt{2}+1)\left(\cos x - \frac{1}{\sqrt{2}}\right)$

$$\Rightarrow (2\cos^2 x - 1) - \frac{(\sqrt{2}+1)}{\sqrt{2}}(\sqrt{2}\cos x - 1) = 0$$

$$\Rightarrow (\sqrt{2}\cos x + 1)(\sqrt{2}\cos x - 1) - \frac{(\sqrt{2}+1)}{\sqrt{2}}(\sqrt{2}\cos x - 1) = 0$$

$$\Rightarrow (\sqrt{2}\cos x - 1)\left\{(\sqrt{2}\cos x + 1) - \frac{(\sqrt{2}+1)}{\sqrt{2}}\right\} = 0$$

$$\Rightarrow (\sqrt{2}\cos x - 1) = 0 \text{ or } \left(\sqrt{2}\cos x - \frac{1}{\sqrt{2}}\right) = 0$$

$$\Rightarrow \cos x = \frac{1}{\sqrt{2}} \text{ or } \cos x = \frac{1}{2}$$

$$\therefore \cos x = \frac{1}{\sqrt{2}} \Rightarrow \cos x = \cos\frac{\pi}{4} \Rightarrow x = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

12. (a) : We have, $\begin{vmatrix} \cos(A+B) & -\sin(A+B) & \cos 2B \\ \sin A & \cos A & \sin B \\ -\cos A & \sin A & \cos B \end{vmatrix} = 0$

$\Rightarrow \cos^2(A + B) + \sin^2(A + B) + \cos 2B$
$\times (\sin^2 A + \cos^2 A) = 0$

$\Rightarrow \cos 2B + 1 = 0 \;\; \Rightarrow \cos 2B = -1 = \cos\pi$

$\Rightarrow 2B = 2n\pi \pm \pi,\ n \in Z \;\; \Rightarrow B = (2n \pm 1)\frac{\pi}{2}, n \in Z$

13. (b) : We have, $x - y = \frac{\pi}{4}$ and $\cot x + \cot y = 2$

Now, $\cot x + \cot y = 2 \Rightarrow \cot\left(y + \frac{\pi}{4}\right) + \cot y = 2$

$\Rightarrow y = \frac{\pi}{6}$ [$\because$ y is the smallest positive angle]

$\therefore x = y + \frac{\pi}{4} = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$

Hence, $x = \frac{5\pi}{12}$ and $y = \frac{\pi}{6}$

14. (a) : We know that, $\sin x \geq 0$, when $0 \leq x \leq \pi$

$\therefore$ $|\sin x| = \sin x$

Hence, $|\sin x| = \sin x + 3 \Rightarrow \sin x = \sin x + 3$ which is impossible.

So, the given equation has no solution in $0 \leq x \leq \pi$

Again $\sin x < 0$, when $\pi < x \leq 2\pi$ $\therefore$ $|\sin x| = -\sin x$

Hence, $|\sin x| = \sin x + 3 \Rightarrow -\sin x = \sin x + 3$

$\Rightarrow \sin x = -\frac{3}{2} < -1$, this is impossible.

Hence, the given equation has no solution in $[0, 2\pi]$

15. (d) : Given, $\tan\theta = \cot 5\theta \Rightarrow \tan\theta - \cot 5\theta = 0$

$\Rightarrow \sin\theta \sin 5\theta - \cos\theta \cos 5\theta = 0 \Rightarrow \cos 6\theta = 0$

$$\Rightarrow \quad \theta = \pm\frac{\pi}{12}, \pm\frac{3\pi}{12}, \pm\frac{5\pi}{12}$$

Now, $\sin 2\theta = \cos 4\theta \Rightarrow \sin 2\theta = 1 - 2\sin^2 2\theta$

$\Rightarrow \quad 2\sin^2 2\theta + \sin 2\theta - 1 = 0$

$\Rightarrow \quad (2\sin 2\theta - 1)(\sin 2\theta + 1) = 0$

$$\Rightarrow \quad \sin 2\theta = \frac{1}{2} \text{ or } \sin 2\theta = -1$$

So, the common values of θ are $-\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$

16. (c) : We know, Area $= \frac{1}{2}ab\sin C$

$$\Rightarrow \quad 15\sqrt{3} = \frac{1}{2}\times 6\times 10\times \sin C$$

$$\Rightarrow \quad \sin C = \frac{\sqrt{3}}{2} \Rightarrow \angle C = 120^\circ$$

Now, $c^2 = a^2 + b^2 - 2ab\cos C$

$$\Rightarrow \quad c^2 = 36 + 100 + 120\times\frac{1}{2} = 196 \Rightarrow c = 14$$

So, $2s = a + b + c \Rightarrow 2s = 30 \Rightarrow s = 15$

$$\therefore \quad r = \frac{\Delta}{s} = \frac{15\sqrt{3}}{15} = \sqrt{3} \Rightarrow r^2 = 3$$

17. (c) : We are given that $3\tan A = \tan B$

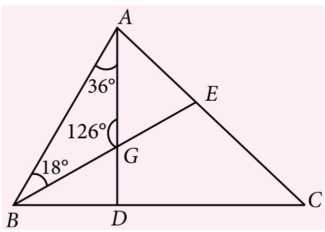
$\Rightarrow \quad 3\sin A\cos B = \sin B\cos A$

$$\Rightarrow \quad \frac{3}{4} = \sin B\cos A \qquad \left[\because \sin A\cos B = \frac{1}{4}\right]$$

$$\text{Now, } \sin A\cos B + \sin B\cos A = \frac{1}{4} + \frac{3}{4}$$

$$\Rightarrow \quad \sin(A+B) = 1 \Rightarrow \angle A + B = \frac{\pi}{2} \Rightarrow \angle C = \frac{\pi}{2}$$

18. (b) : In the triangle ΔAGB, we have



$$\frac{AG}{\sin 18^\circ} = \frac{AB}{\sin 126^\circ}$$

$$\Rightarrow \quad AB = \frac{\sin 126^\circ}{\sin 18^\circ}AG$$

$$\Rightarrow \quad AB = \frac{\cos 36^\circ}{\sin 18^\circ}\times\frac{2}{3}AD$$

$$\Rightarrow \quad AB = \frac{\sqrt{5}+1}{\sqrt{5}-1}\times\frac{2}{3}\times 6 = 2(3+\sqrt{5})$$

$$\text{Now, } R = \frac{c}{2\sin C} \Rightarrow R = (3+\sqrt{5})\operatorname{cosec} C$$

19. (c) : Let ABC be the isosceles triangle with $AB = AC$ and $\angle A = \frac{2\pi}{3}$.

$$\text{Now, } \Delta = \frac{1}{2}(AB\times AC)\times\sin\frac{2\pi}{3}$$

$$\Rightarrow \quad \Delta = \frac{\sqrt{3}}{4}x^2, \text{ where } AB = AC = x\text{(say)}$$

$$\text{Now, } \frac{AB}{\sin 30^\circ} = \frac{BC}{\sin 120^\circ} = \frac{AC}{\sin 30^\circ} \Rightarrow 2x = \frac{2a}{\sqrt{3}}$$

$$\Rightarrow x = \frac{a}{\sqrt{3}} (a = BC).$$

$$\therefore \quad 2s = 2x + a \Rightarrow 2s = \frac{2a}{\sqrt{3}} + a$$

$$\Rightarrow \quad s = \left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)a \text{ and } \Delta = \frac{\sqrt{3}}{4}\times\frac{a^2}{3} = \frac{a^2}{4\sqrt{3}}$$

$$\text{Now, } r = \frac{\Delta}{s} \Rightarrow \sqrt{3} = \frac{a^2}{4\sqrt{3}}\times\frac{2\sqrt{3}}{(2+\sqrt{3})a}$$

$$\Rightarrow a = 6 + 4\sqrt{3} \ \therefore \ \Delta = 12 + 7\sqrt{3}$$

20. (b) : Let ABC be the triangle such that its sides $a = BC$ and $b = CA$ are the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$

$$\therefore \quad a + b = 2\sqrt{3} \text{ and } ab = 2.$$

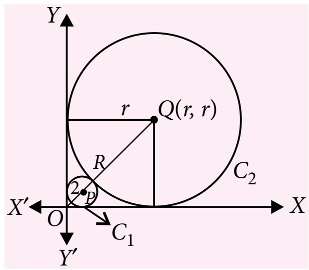
$$\text{Also, } \angle C = \frac{\pi}{3} \Rightarrow \cos C = \frac{1}{2}$$

$$\Rightarrow \quad \frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2} \Rightarrow (a+b)^2 - c^2 = 3ab$$

$$\Rightarrow \quad (2\sqrt{3})^2 - c^2 = 3\times 2 \Rightarrow c^2 = 6 \Rightarrow c = \sqrt{6}$$

$$\therefore \quad \text{Perimeter of } \Delta ABC = a + b + c = 2\sqrt{3} + \sqrt{6}$$

21. (a) : We observe that



$OQ = OR + RQ$

$\Rightarrow \quad \sqrt{2}r = OP + PR + RQ$

$\Rightarrow \quad \sqrt{2}r = 2\sqrt{2} + 2 + r$

$$\Rightarrow \quad r = 2\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$= 2(3 + 2\sqrt{2}) = 6 + 4\sqrt{2}$

22. (a)

23. (a) : The equation of the given circle is

$x^2 + y^2 - 6x - 10y + \lambda = 0$

Given that the circle does not cut the coordinate axes

$\therefore$ r (radius) $< 3 \Rightarrow (3)^2 + (5)^2 - \lambda < 9 \Rightarrow \lambda > 25$ ...(i)

Again, the point (2, 4) lies inside the circle

$\therefore \quad 4 + 16 - 12 - 40 + \lambda < 0 \Rightarrow \lambda < 32$ ...(ii)

Now from (i) and (ii), we get $\lambda \in (25, 32)$.

24. (b) : The given equation of the circle is $x^2 + y^2 - x + 3y = 0$. Centre and radius are $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and $\sqrt{\frac{5}{2}}$ respectively.

Let $y = mx$ be the equation of L_1.
Then, p_1 = Length of the intercept on L_1.

$$\Rightarrow \; p_1 = 2\sqrt{\left(\sqrt{\frac{5}{2}}\right)^2 - \left(\frac{m+3}{2\sqrt{m^2+1}}\right)^2}$$

$$\Rightarrow \; p_1 = 2\sqrt{\frac{5}{2} - \frac{(m+3)^2}{4(m^2+1)}}$$

And p_2 = Length of the intercept on L_2.

$$\Rightarrow \; p_2 = 2\sqrt{\left(\sqrt{\frac{5}{2}}\right)^2 - (\sqrt{2})^2} \;\Rightarrow\; p_2 = 2\sqrt{\frac{5}{2} - 2} = \sqrt{2}$$

Now, $p_1 = p_2 \Rightarrow 2\sqrt{\frac{5}{2} - \frac{(m+3)^2}{4(m^2+1)}} = \sqrt{2}$

$\Rightarrow \; 7m^2 - 6m - 1 = 0 \quad \Rightarrow \quad m = 1, -\frac{1}{7}$

So, the equations of L_1 are $y = x$ and $7y = -x$

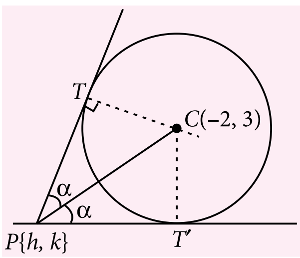
25. (a) : Here, centre of the given circle is the point of intersection of the normals at $A(-3, 0)$ and $B(4, 1)$.
The equation of a line through $(-3, 0)$ and perpendicular to $4x - 3y = -12$ is

$y - 0 = -\frac{3}{4}(x + 3) \;\Rightarrow\; 3x + 4y + 9 = 0$ ...(i)

Similarly, the equation of a line through B (4, 1) and perpendicular to $3x + 4y = 16$ is $y - 1 = \frac{4}{3}(x - 4)$
$\Rightarrow \; 4x - 3y = 13 = 0$ ...(ii)
Solving (i) and (ii) we get $x = 1, y = -3$. So, the centre is $C(1, -3)$.
$\therefore$ Radius $= CA = \sqrt{16 + 9} = 5$
Hence the equation of the required circle is
$(x - 1)^2 + (y + 3)^2 = 5^2$ or $x^2 + y^2 - 2x + 6y - 15 = 0$

26. (d) : The equation of the circle is
$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$
Its centre (C) $\cong (-2, 3)$
and radius $= \sqrt{4 + 9 - 9\sin^2\alpha - 13\cos^2\alpha} = 2\sin\alpha$

Let the coordinates of P be (h, k).
Clearly, CP bisects $\angle TPT' = 2\alpha$
$\therefore \angle CPT = \angle CPT' = \alpha$
Now, in ΔCPT, we have



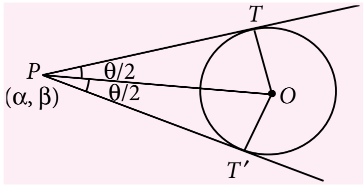
$\sin\alpha = \frac{CT}{CP}$

$$\Rightarrow \sin\alpha = \frac{2\sin\alpha}{\sqrt{(h+2)^2 + (k-3)^2}}$$

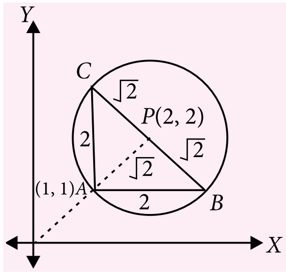
$\Rightarrow \; (h+2)^2 + (k-3)^2 = 4 \;\Rightarrow\; h^2 + k^2 + 4h - 6k + 9 = 0$
Hence, the locus of (h, k) is $x^2 + y^2 + 4x - 6y + 9 = 0$

27. (b) : Let PT and PT' be the tangents drawn from $P(\alpha, \beta)$ to the circle $x^2 + y^2 = a^2$, and let $\angle TPT' = \theta$. If O is the centre of the circle, then $\angle TPO = \angle T'PO = \frac{\theta}{2}$.

$\therefore \; \tan\frac{\theta}{2} = \frac{OT}{TP} = \frac{a}{\sqrt{s_1}} \Rightarrow \theta = 2\tan^{-1}\left(\frac{a}{\sqrt{s_1}}\right)$



28. (b) : Since the chord BC is perpendicular to AP and passes through the centre $P(2, 2)$, so its equation is $x + y = 4$



29. (a) : It is given that the difference in the radii of two concentric circles is d. So, the equations of two circles are $x^2 + y^2 = 4$ and $x^2 + y^2 = (2 + d)^2$
If the line $y = x + 1$ cuts the circles in real points, then $x^2 + (x + 1)^2 = (2 + d)^2$ must have real roots
$\therefore \; 4 - 8 + 8(2 + d)^2 > 0 \;\Rightarrow\; 2(2 + d)^2 - 1 > 0$

$$\Rightarrow \; (2+d)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 > 0$$

$$\Rightarrow \; \left(2 + d - \frac{1}{\sqrt{2}}\right)\left(2 + d + \frac{1}{\sqrt{2}}\right) > 0$$

$$\Rightarrow \; d \in \left(-\infty, -2 - \frac{1}{\sqrt{2}}\right) \cup \left(-2 + \frac{1}{\sqrt{2}}, \infty\right)$$

30. (c) : The area of the circle $= \pi r^2 = 49\pi \Rightarrow r = 7$.
Also the centre of the circle is the point of intersection of the diameters $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ solving the above equation we get centre $(1, -1)$.
$\therefore$ Equation of circle is $(x - 1)^2 + (y + 1)^2 = 49$
$\Rightarrow \; x^2 + y^2 - 2x + 2y - 47 = 0$

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BRAIN@WORK

TRIGONOMETRIC RATIOS AND IDENTITIES

SOME IMPORTANT FORMULAE

1. Addition Formulae

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iii) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

2. Subtraction Formulae

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(ii) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

3. Multiple Angle Formulae

(i) Functions of 2*A*

(a) $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(c) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(ii) Functions of 3*A*

(a) $\sin 3A = 3\sin A - 4\sin^3 A$

(b) $\cos 3A = 4\cos^3 A - 3\cos A$

(c) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

4. Expressing Products of Trigonometric Functions as Sum or Difference

(i) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

(iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

(iv) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

The above four formulae can be obtained by expanding the right hand side and then simplifying.

5. Expressing Sum or Difference of two sines or two cosines as a product

In the formulae derived in the earlier section, if we put $A + B = C$ and $A - B = D$, then $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$, these formulae can be rewritten as

(i) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}.\cos\dfrac{C-D}{2}$

(ii) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\cdot\sin\dfrac{C-D}{2}$

(iii) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cdot\cos\dfrac{C-D}{2}$

(iv) $\cos D - \cos C = 2\sin\dfrac{C+D}{2}\cdot\sin\dfrac{C-D}{2}$

6. Some more results

(i) $\sin(A + B) \times \sin(A - B) = \sin^2 A - \sin^2 B$
$= \cos^2 B - \cos^2 A$

(ii) $\cos(A + B) \times \cos(A - B) = \cos^2 A - \sin^2 B$
$= \cos^2 B - \sin^2 A$

(iii) $\sin 18° = \dfrac{\sqrt{5}-1}{4} = \cos 72°$

(iv) $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$

(v) $\cos 36° = \dfrac{\sqrt{5}+1}{4} = \sin 54°$

(vi) $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$

(vii) $\tan 22\dfrac{1}{2}° = \sqrt{2} - 1$

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TRIPLE ANGLE FORMULAE

1. $\sin\theta \sin(60° - \theta) \sin(60° + \theta) = \frac{1}{4}\sin 3\theta$
2. $\cos\theta \cos(60° - \theta) \cos(60° + \theta) = \frac{1}{4}\cos 3\theta$
3. $\tan\theta \tan(60° - \theta) \tan(60° + \theta) = \tan 3\theta$

CONDITIONAL TRIGONOMETRICAL IDENTITIES

If $A + B + C = \pi$, then

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
2. $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$
3. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
4. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$
5. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$
6. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
7. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$
8. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

SOME OTHER USEFUL RESULTS

1. $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots$ to n terms
$$= \frac{\sin[\alpha + ((n-1)\beta/2)]\sin(n\beta/2)}{\sin(\beta/2)}$$
2. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots$ to n terms
$$= \frac{\cos[\alpha + ((n-1)\beta/2)]\sin(n\beta/2)}{\sin(\beta/2)}$$
3. $\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1 + \sin A}$
4. $\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1 - \sin A}$
5. $\sin A \pm \cos A = \sqrt{2}\sin\left(\frac{\pi}{4} \pm A\right) = \sqrt{2}\cos\left(A \mp \frac{\pi}{4}\right)$
6. $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$
7. $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$

TRIGONOMETRICAL RATIOS OF MORE THAN THREE ANGLES

1. $\sin(A_1 + A_2 + A_3 + \ldots + A_n) = \cos A_1 \cdot \cos A_2 \cdot \ldots \cdot \cos A_n \times (S_1 - S_3 + S_5 - \ldots)$
2. $\cos(A_1 + A_2 + A_3 + \ldots + A_n) = \cos A_1 \cdot \cos A_2 \cdot \ldots \cdot \cos A_n \times (1 - S_2 + S_4 - \ldots)$
3. $\tan(A_1 + A_2 + A_3 + \ldots + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$

where $S_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n$
= Sum of tangent angles taken one at a time.
$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \ldots + \tan A_{n-1} \tan A_n$
= Sum of tangent angles taken two at a time.
$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots$
= Sum of tangent angles taken three at a time and so on.
If $A_1 = A_2 = A_3 = \ldots = A_n = A$,
then
$S_1 = {}^nC_1 \tan A,\quad S_2 = {}^nC_2 \tan^2 A,\quad S_3 = {}^nC_3 \tan^3 A, \ldots$ and so on.

- $\sin nA = \cos^n A(S_1 - S_3 + S_5 - S_7 - \ldots)$
- $\cos nA = \cos^n A(1 - S_2 + S_4 - S_6 + \ldots)$
- $\tan nA = \dfrac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$
- $\cos A \cdot \cos 2A \cdot \cos 4A \cdot \ldots \cdot \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

Bounds of the Expression $a \cos\theta + b \sin\theta$

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right)$$
$$= \sqrt{a^2+b^2}\,(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$
$$= \sqrt{a^2+b^2}\,\sin(\theta + \alpha), \text{ where } \tan\alpha = \frac{a}{b}$$

Also, $a \cos\theta + b \sin\theta = \sqrt{a^2+b^2}\cos(\theta - \beta)$, where $\tan\beta = \frac{b}{a}$

Since, $-1 \le \sin(\theta + \alpha) \le 1$

Hence, $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

EXERCISE

1. If $\sin\alpha$, $\sin\beta$ and $\cos\alpha$ are in G.P, then roots of the equation $x^2 + 2x\cot\beta + 1 = 0$ are always

(a) equal (b) real
(c) imaginary (d) greater than 1

2. The value of $\sin\frac{\pi}{18}\cdot\sin\frac{5\pi}{18}\cdot\sin\frac{7\pi}{18}$ is equal to

(a) $\frac{1}{8}$ (b) $\frac{1}{16}$
(c) $\frac{1}{4}$ (d) none of these

3. If ABC is a triangle such that angle A is obtuse, then
(a) $\tan B \tan C > 1$ (b) $\tan B \tan C < 1$
(c) $\tan B \tan C = 1$ (d) none of these

4. If in a triangle ABC, $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in A.P, then $\cos A$, $\cos B$, $\cos C$ are in
(a) H.P (b) A.P
(c) G.P (d) none of these

5. If $\sin\alpha \sin\beta - \cos\alpha \cos\beta + 1 = 0$, then the value of $\cot\alpha \cdot \tan\beta$ is
(a) -1 (b) 0
(c) 1 (d) none of these

6. $\frac{\sin 5\theta + \sin 2\theta - \sin\theta}{\cos 5\theta + 2\cos 3\theta + 2\cos^2\theta + \cos\theta}$ is equal to
(a) $\tan\theta$ (b) $\cos\theta$
(c) $\cot\theta$ (d) none of these

7. If α and β are solutions of $\sin^2 x + a(\sin x) + b = 0$ as well as that of $\cos^2 x + c(\cos x) + d = 0$, then $\sin(\alpha + \beta)$ is equal to
(a) $\frac{2bd}{b^2+d^2}$ (b) $\frac{a^2+c^2}{2ac}$
(c) $\frac{b^2+d^2}{2bd}$ (d) $\frac{2ac}{a^2+c^2}$

8. If $a\sin^2\theta + b\cos^2\theta = m$, $b\sin^2\phi + a\cos^2\phi = n$ and $a\tan\theta = b\tan\phi$, then $\frac{1}{m} + \frac{1}{n}$ is equal to
(a) $\frac{1}{a} - \frac{1}{b}$ (b) $\frac{1}{a} + \frac{1}{b}$
(c) $\frac{1}{a}$ (d) $\frac{1}{b}$

9. In a ΔABC, if $\cot A \cot B \cot C > 0$, then the triangle is
(a) acute angled (b) right angled
(c) obtuse angled (d) does not exist

10. If in a ΔABC, $\sin^2 A + \sin^2 B + \sin^2 C = 2$, then the triangle is always
(a) isosceles triangle (b) right angled
(c) acute angled (d) obtuse angled

11. The number of solutions of the equation $\cos^{-1}(1 - x) + m\cos^{-1}x = \frac{n\pi}{2}$, where $m > 0$, $n \le 0$, is
(a) 0 (b) 1
(c) 2 (d) none of these

12. If $m \cdot \sin(\alpha + \beta) = \cos(\alpha - \beta)$, then $\frac{1}{1 - m\sin 2\alpha} + \frac{1}{1 - m\sin 2\beta}$ is equal to
(a) $\frac{2}{1-m^2}$ (b) $\frac{2}{m^2-1}$
(c) $\frac{3}{1-m^2}$ (d) $\frac{3}{m^2-1}$

13. $\cos\frac{\pi}{8}\cos\frac{3\pi}{8}\cos\frac{5\pi}{8}\cos\frac{7\pi}{8}$ is equal to
(a) $\frac{1}{2}$ (b) $\frac{1-\sqrt{2}}{2\sqrt{2}}$ (c) $\frac{1}{8}$ (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

14. The minimum value of $\cos(\cos x)$ is
(a) 0 (b) $-\cos 1$ (c) $\cos 1$ (d) -1

15. If $\cos\alpha = \frac{2\cos\beta - 1}{2 - \cos\beta}, (0 < \alpha < \pi, 0 < \beta < \pi)$, then $\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$ is equal to
(a) 1 (b) $\sqrt{2}$
(c) $\sqrt{3}$ (d) none of these

16. The maximum value of $\cos x(\sin x + \cos x)$ is
(a) 2 (b) $\sqrt{2}$
(c) 1 (d) none of these

17. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$, then $x + y + z$ is
(a) 0 (b) 1
(c) -1 (d) none of these

18. If $\pi < 2\theta < \frac{3\pi}{2}$, then $\sqrt{2 + \sqrt{2 + 2\cos 4\theta}}$ equals to
(a) $-2\cos\theta$ (b) $-2\sin\theta$ (c) $2\cos\theta$ (d) $2\sin\theta$

19. If $\sin\alpha + \sin\beta = a$, $\cos\alpha + \cos\beta = b$, then $\tan\frac{\alpha - \beta}{2}$ is equal to
(a) $\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$ (b) $-\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$
(c) Both (a) and (b) (d) none of these

20. If $\cos(x - y)\cos(z - t) = \cos(x + y)\cos(z + t)$, then $\tan x \tan y + \tan z \tan t$ is equal to
(a) 1 (b) -1 (c) 2 (d) 0

21. If $\sin\theta = 3\sin(\theta + 2\alpha)$, then the value of $\tan(\theta + \alpha) + 2\tan\alpha$ is
(a) 3 (b) 2 (c) 1 (d) 0

22. Maximum value of the expression $2\sin x + 4\cos x + 3$ is

(a) $2\sqrt{5}+3$ (b) $2\sqrt{5}-3$

(c) $\sqrt{5}+3$ (d) none of these

23. If $\sin x = \cos^2 x$, then $\cos^2 x\,(1 + \cos^2 x)$ equals to

(a) 0 (b) 1

(c) 2 (d) none of these

24. If $\tan\theta\tan\alpha = \sqrt{\dfrac{a-b}{a+b}}$, then the value of $(a - b\cos 2\alpha)(a - b\cos 2\theta)$ is

(a) $(a^2 - b^2)\tan\alpha$ (b) $(a^2 - b^2)\tan\theta$

(c) $a^2 - b^2$ (d) $a^2 + b^2$

25. If $(\tan x - \tan y)^2$, $(\tan y - \tan z)^2$ and $(\tan z - \tan x)^2$ are in A.P, then $(\tan x - \tan y)$, $(\tan y - \tan z)$ and $(\tan z - \tan x)$ are in

(a) A.P (b) G.P

(c) H.P (d) none of these

26. If $\sin(x-y) = \cos(x+y) = \dfrac{1}{2}$, then the values of x and y lying between 0° and 180° are given by

(a) $x = 45°, y = 45°$ (b) $x = 45°, y = 135°$

(c) $x = 165°, y = 15°$ (d) $x = 165°, y = 135°$

27. $\dfrac{1}{\sin 3\alpha}\left[\sin^3\alpha + \sin^3\left(\dfrac{2\pi}{3}+\alpha\right) + \sin^3\left(\dfrac{4\pi}{3}+\alpha\right)\right]$ is equal to

(a) $\dfrac{4}{3}$ (b) $\dfrac{3}{4}$

(c) $-\dfrac{3}{4}$ (d) none of these

28. If $\alpha, \beta \in [0, \pi]$, then minimum value of $\sin\left(\dfrac{\alpha+\beta}{2}\right)$ is

(a) $\dfrac{\sin\alpha+\sin\beta}{2}$ (b) $|\sin\alpha - \sin\beta|$

(c) $\dfrac{\cos\alpha+\cos\beta}{2}$ (d) $|\cos\alpha - \cos\beta|$

29. If $x, \dfrac{\pi}{4}, y$ are in A.P, then the value of $\tan x \tan\dfrac{\pi}{4}\tan y$ is

(a) 1 (b) –1

(c) 0 (d) none of these

30. If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$, then which of the following is true?

(a) $\sin\theta\cos\theta = \dfrac{1}{x}$ (b) $\sin\theta\tan\theta = y$

(c) $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$

(d) all the above

31. In a triangle ABC, $\tan\dfrac{A}{2}, \tan\dfrac{B}{2}, \tan\dfrac{C}{2}$ are H.P. then the minimum value of $\cot\dfrac{B}{2}$ is

(a) $3\sqrt{3}$ (b) $\sqrt{3}$

(c) 3 (d) none of these

32. If $(m + 2)\sin\theta + (2m - 1)\cos\theta = 2m + 1$, then

(a) $\tan\theta = \dfrac{3}{4}$ (b) $\tan\theta = \dfrac{4}{5}$

(c) $\tan\theta = \dfrac{2m}{m^2-1}$ (d) $\tan\theta = \dfrac{2m}{m^2+1}$

33. If $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = k\sin^2\left(\dfrac{\alpha-\beta}{2}\right)$ then k is equal to

(a) 4 (b) 2 (c) 1 (d) 3

34. If $x = \sec\phi - \tan\phi$ and $y = \operatorname{cosec}\phi + \cot\phi$, then

(a) $x = \dfrac{y+1}{y-1}$ (b) $x = \dfrac{1-y}{y+1}$

(c) $y = \dfrac{1-x}{1+x}$ (d) $xy + x - y + 1 = 0$

35. The value of $\log\cot 1° + \log\cot 2° + \log\cot 3° + \ldots + \log\cot 89°$ is

(a) 0 (b) 1 (c) 1/2 (d) 3/4

SOLUTIONS

1. (b) : Since, $\sin\alpha, \sin\beta, \cos\alpha$ are in G.P.

$\Rightarrow$ $\sin^2\beta = \sin\alpha\cos\alpha \Rightarrow \cos 2\beta = 1 - \sin 2\alpha \ge 0$

Now, the discriminant of the given equation is

$4\cot^2\beta - 4 = 4\cos 2\beta \cdot \operatorname{cosec}^2\beta \ge 0$

$\Rightarrow$ Roots are always real.

2. (a) : $\cos\left(\dfrac{\pi}{2}-\dfrac{\pi}{18}\right)\cdot\cos\left(\dfrac{\pi}{2}-\dfrac{5\pi}{18}\right)\cdot\cos\left(\dfrac{\pi}{2}-\dfrac{7\pi}{18}\right)$

$$= \cos\frac{4\pi}{9}\cdot\cos\frac{2\pi}{9}\cdot\cos\frac{\pi}{9} = \frac{\sin\left(2^3\dfrac{\pi}{9}\right)}{2^3\cdot\sin\dfrac{\pi}{9}} = \frac{\sin\dfrac{8\pi}{9}}{8\sin\dfrac{\pi}{9}} = \frac{1}{8}$$

3. (b) : $\tan A = -\tan(B + C)$

$\Rightarrow \tan A = \dfrac{\tan B + \tan C}{\tan B\tan C - 1}$

Since A is obtuse,

$\therefore$ $\tan B\tan C - 1 < 0 \Rightarrow \tan B\tan C < 1$.

4. (b) : Since $\tan\dfrac{A}{2}, \tan\dfrac{B}{2}, \tan\dfrac{C}{2}$ are in A.P,

$\therefore\ 2\tan\dfrac{B}{2} = \tan\dfrac{A}{2} + \tan\dfrac{C}{2}$

$$\Rightarrow \frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{B+C}{2}\right)} = \frac{\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}$$

$$\Rightarrow \sin\frac{A-B}{2}\sin\frac{A+B}{2} = \sin\frac{B-C}{2}\sin\frac{B+C}{2}$$

$\Rightarrow$ $\cos B - \cos A = \cos C - \cos B$

$\Rightarrow$ $\cos A, \cos B, \cos C$ are in A.P.

5. (a) : Given, $\sin\alpha \sin\beta - \cos\alpha \cos\beta + 1 = 0$

$\Rightarrow$ $\cos(\alpha + \beta) = 1 \Rightarrow \alpha + \beta = 2n\pi$

$\Rightarrow$ $\sin(\alpha + \beta) = 0 \Rightarrow \sin\alpha \cos\beta + \cos\alpha \sin\beta = 0$

$\Rightarrow$ $\cot\alpha \tan\beta = -1$.

6. (a) : $$\frac{2\sin 2\theta\cos 3\theta + \sin 2\theta}{2\cos 3\theta\cos 2\theta + 2\cos 3\theta + 2\cos^2\theta}$$

$$= \frac{\sin 2\theta[2\cos 3\theta + 1]}{2[\cos 3\theta(\cos 2\theta + 1) + (\cos^2\theta)]}$$

$$= \frac{\sin 2\theta[2\cos 3\theta + 1]}{2[\cos 3\theta(2\cos^2\theta) + \cos^2\theta]}$$

$$= \frac{\sin 2\theta(2\cos 3\theta + 1)}{2\cos^2\theta(2\cos 3\theta + 1)} = \tan\theta$$

7. (d) : According to the given condition,
$\sin\alpha + \sin\beta = -a$ and $\cos\alpha + \cos\beta = -c$.

$$\Rightarrow 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -a$$

and $2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -c \Rightarrow \tan\frac{\alpha+\beta}{2} = \frac{a}{c}$

$$\Rightarrow \sin(\alpha+\beta) = \frac{2\tan\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}} = \frac{2ac}{a^2+c^2}$$

8. (b) : $a\tan^2\theta + b = m(1 + \tan^2\theta)$

$$\Rightarrow (a-m)\tan^2\theta = m-b \Rightarrow \tan^2\theta = \frac{m-b}{a-m}$$

$b\tan^2\phi + a = n(1 + \tan^2\phi) \Rightarrow (b - n)\tan^2\phi = n - a$

$$\therefore \tan^2\phi = \frac{n-a}{b-n}$$

We have, $a\tan\theta = b\tan\phi \Rightarrow a^2\frac{m-b}{a-m} = b^2\frac{n-a}{b-n}$

$$\Rightarrow \left(\frac{1}{m}+\frac{1}{n}\right)ab(a-b) = a^2 - b^2$$

$$\Rightarrow \frac{1}{m}+\frac{1}{n} = \frac{1}{a}+\frac{1}{b}.$$

9. (a) : Since $\cot A \cot B \cot C > 0$
$\cot A, \cot B, \cot C$ are positive $\Rightarrow \Delta$ is acute angled

10. (b) : $\sin^2 A + \sin^2 B + \sin^2 C = 2$

$\Rightarrow$ $2\cos A \cos B \cos C = 0$

$\Rightarrow$ Either $A = 90°$ or $B = 90°$ or $C = 90°$

11. (a) : $\cos^{-1}(1 - x)$ is defined if $-1 \le 1 - x \le 1$

$\Rightarrow$ $0 \le x \le 2$

and $\cos^{-1}x$ is defined if $-1 \le x \le 1$

So, $\cos^{-1}(1 - x) + m\cos^{-1}x$ is defined if $0 \le x \le 1$

When $0 \le x \le 1$, also $0 \le 1 - x \le 1$

So, $0 \le \cos^{-1}(1-x) \le \frac{\pi}{2}$ and $0 \le \cos^{-1}x \le \frac{\pi}{2}$

So, L.H.S. ≥ 0 but R.H.S. ≤ 0 if $n \le 0$.

So, equality holds if L.H.S. = R.H.S. = 0

Now, L.H.S. = 0 if $\cos^{-1}(1 - x) = 0$ and $\cos^{-1}x = 0$ which is not possible.

12. (a) : $$\frac{1}{1-m\sin 2\alpha} + \frac{1}{1-m\sin 2\beta}$$

$$= \frac{2-m(\sin 2\alpha + \sin 2\beta)}{1-m(\sin 2\alpha+\sin 2\beta)+m^2\sin 2\alpha\sin 2\beta}$$

$$= \frac{2-2m\sin(\alpha+\beta)\cos(\alpha-\beta)}{1-2m\sin(\alpha+\beta)\cos(\alpha-\beta)+4m^2\sin\alpha\cos\alpha\sin\beta\cos\beta}$$

$$= \frac{2[1-\cos^2(\alpha-\beta)]}{1-2\cos^2(\alpha-\beta)+m^2[\sin(\alpha+\beta)+\sin(\alpha-\beta)][\sin(\alpha+\beta)-\sin(\alpha-\beta)]}$$

$$= \frac{2\sin^2(\alpha-\beta)}{1-2\cos^2(\alpha-\beta)+m^2\sin^2(\alpha+\beta)-m^2\sin^2(\alpha-\beta)}$$

$$= \frac{2\sin^2(\alpha-\beta)}{1-\cos^2(\alpha-\beta)-m^2\sin^2(\alpha-\beta)} = \frac{2}{1-m^2}$$

13. (c) : $$\cos\frac{\pi}{8}\cdot\cos\frac{3\pi}{8}\cdot\cos\frac{5\pi}{8}\cdot\cos\frac{7\pi}{8}$$

$$= \cos\frac{\pi}{8}\cdot\sin\frac{\pi}{8}\cdot\left(-\sin\frac{\pi}{8}\right)\cdot\left(-\cos\frac{\pi}{8}\right)$$

$$= \frac{1}{4}\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2 = \frac{1}{4}\sin^2\frac{\pi}{4} = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}.$$

14. (c) : $\cos x$ varies from -1 to 1 for all real x.
Thus $\cos(\cos x)$ varies from $\cos 1$ to $\cos 0$

$\Rightarrow$ Minimum value of $\cos(\cos x)$ is $\cos 1$.

15. (c) : Given, $\cos\alpha = \frac{2\cos\beta - 1}{2-\cos\beta}$

$$\Rightarrow \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2\left(1-\tan^2\frac{\beta}{2}\right)-\left(1+\tan^2\frac{\beta}{2}\right)}{2\left(1+\tan^2\frac{\beta}{2}\right)-\left(1-\tan^2\frac{\beta}{2}\right)}$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-3\tan^2\frac{\beta}{2}}{1+3\tan^2\frac{\beta}{2}}$$

Applying componendo and dividendo, we have,

$$\tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2} \Rightarrow \tan\frac{\alpha}{2}\cdot\cot\frac{\beta}{2} = \sqrt{3}$$

16. (d) : $\cos x(\sin x + \cos x) = \frac{1}{2}(\sin 2x + \cos 2x + 1)$

$$= \frac{1}{2}\left(\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)+1\right)$$

$\therefore$ Maximum value $= \frac{\sqrt{2}+1}{2}$

17. (a) : Let $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta-\frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta+\frac{2\pi}{3}\right)} = k(\text{say})$

$$\therefore \quad x+y+z = k\left[\cos\theta+\cos\left(\theta-\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)\right]$$

$$\Rightarrow \quad x+y+z = k\left[\cos\theta + 2\cos\theta\cdot\cos\left(\frac{2\pi}{3}\right)\right] = 0$$

18. (d) : $\sqrt{2+\sqrt{2(1+\cos 4\theta)}} = \sqrt{2+2|\cos 2\theta|}$

$$= \sqrt{2(1-\cos 2\theta)}$$

$$= 2|\sin\theta| = 2\sin\theta \quad \left(\because \frac{\pi}{2} < \theta < \frac{3\pi}{4}\right)$$

19. (c) : Given, $\sin\alpha + \sin\beta = a$

$$\Rightarrow \quad 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = a$$

Also, $\cos\alpha + \cos\beta = b$

$$\Rightarrow \quad 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = b$$

Hence, $\tan\frac{\alpha+\beta}{2} = \frac{a}{b} \Rightarrow \sec^2\frac{\alpha+\beta}{2} = 1+\frac{a^2}{b^2}$

or $\cos\frac{\alpha+\beta}{2} = \frac{b}{\sqrt{a^2+b^2}} \Rightarrow \cos\frac{\alpha-\beta}{2} = \pm\frac{\sqrt{a^2+b^2}}{2}$

$$\Rightarrow \quad \tan\frac{\alpha-\beta}{2} = \pm\sqrt{\sec^2\frac{\alpha-\beta}{2}-1} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

20. (d) : $\cos(x-y)\cos(z-t) = \cos(x+y)\cos(z+t)$

$$\Rightarrow \quad \frac{\cos(x-y)}{\cos(x+y)} = \frac{\cos(z+t)}{\cos(z-t)}$$

$$\Rightarrow \quad \frac{\cos(x+y)+\cos(x-y)}{\cos(x+y)-\cos(x-y)} = \frac{\cos(z+t)+\cos(z-t)}{\cos(z-t)-\cos(z+t)}$$

$$\Rightarrow \quad \frac{2\cos(x)\cos(y)}{2\sin(x)\sin(-y)} = \frac{2\cos z\cos t}{2\sin z\sin t}$$

$\therefore$ $\tan x \tan y + \tan z \tan t = 0.$

21. (d) : Given, $\sin\theta = 3\sin(\theta + 2\alpha)$

$\Rightarrow$ $\sin(\theta + \alpha - \alpha) = 3\sin(\theta + \alpha + \alpha)$

$\Rightarrow$ $\sin(\theta + \alpha)\cos\alpha - \cos(\theta + \alpha)\sin\alpha$

$= 3\sin(\theta + \alpha)\cos\alpha + 3\cos(\theta + \alpha)\sin\alpha$

$\Rightarrow$ $-2\sin(\theta + \alpha)\cos\alpha = 4\cos(\theta + \alpha)\sin\alpha$

$$\Rightarrow \quad \frac{-\sin(\theta+\alpha)}{\cos(\theta+\alpha)} = \frac{2\sin\alpha}{\cos\alpha} \Rightarrow \tan(\theta+\alpha)+2\tan\alpha = 0$$

22. (a) : Maximum value of $2\sin x + 4\cos x = 2\sqrt{5}$ Hence the maximum value of $2\sin x + 4\cos x + 3$ is $2\sqrt{5}+3$.

23. (b) : We have, $\cos^2 x\,(1 + \cos^2 x) = \cos^2 x + \cos^4 x$

$= \cos^2 x + \sin^2 x = 1$

24. (c) : We have, $\tan^2\theta\tan^2\alpha = \frac{a-b}{a+b}$

$\Rightarrow$ $(a + b)\tan^2\theta = (a - b)\cot^2\alpha$

and $(a + b)\tan^2\alpha = (a - b)\cot^2\theta$

Now, $(a - b\cos2\theta)(a - b\cos2\alpha)$

$$= \left[a-b\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\right]\left[a-b\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)\right]$$

$$= \left\{\frac{(a-b)+(a+b)\tan^2\theta}{1+\tan^2\theta}\right\}\left\{\frac{(a-b)+(a+b)\tan^2\alpha}{1+\tan^2\alpha}\right\}$$

$$= \left\{\frac{(a-b)+(a-b)\cot^2\alpha}{1+\tan^2\theta}\right\}\left\{\frac{(a-b)+(a-b)\cot^2\theta}{1+\tan^2\alpha}\right\}$$

$$= (a-b)^2\left(\frac{1+\cot^2\alpha}{1+\tan^2\theta}\right)\left(\frac{1+\cot^2\theta}{1+\tan^2\alpha}\right)$$

$$= (a-b)^2\left(\frac{1+\tan^2\alpha}{1+\tan^2\theta}\right)\left(\frac{1+\tan^2\theta}{1+\tan^2\alpha}\right)\times\frac{1}{\tan^2\theta\tan^2\alpha}$$

$$= \frac{(a-b)^2}{\left(\frac{a-b}{a+b}\right)} = a^2 - b^2$$

25. (c) : Let $a = \tan x - \tan y$, $b = \tan y - \tan z$ and $c = \tan z - \tan x$

$\therefore$ $a + b + c = 0$ ...(i)

From (i), $b^2 = a^2 + c^2 + 2ac$ ...(ii)
According to question, $2b^2 = a^2 + c^2$ ...(iii)
$\Rightarrow \quad 2b^2 = b^2 - 2ac$ [Using (ii)]
$\Rightarrow -b^2 = 2ac$

$$\Rightarrow \quad -b = \frac{2ac}{b} = \frac{2ac}{-(a+c)} \quad \Rightarrow \quad b = \frac{2ac}{a+c}$$

$\therefore$ a, b, c are in H.P.

26. (d) : $\sin(x-y) = \frac{1}{2} \Rightarrow x - y = 30°$ or $150°$

$\cos(x+y) = \frac{1}{2} \Rightarrow x + y = 60°$ or $300°$

Only option (d) satisfies the above two equations
$\therefore$ $x = 165°$ and $y = 135°$

27. (c) $\frac{1}{\sin 3\alpha}\left[\sin^3\alpha + \sin^3\left(\frac{2\pi}{3}+\alpha\right) + \sin^3\left(\frac{4\pi}{3}+\alpha\right)\right]$

$$= \frac{1}{4\sin 3\alpha}\left[3\sin\alpha - \sin 3\alpha + 3\sin\left(\frac{2\pi}{3}+\alpha\right) - \sin(2\pi + 3\alpha) + 3\sin\left(\frac{4\pi}{3}+\alpha\right) - \sin(4\pi + 3\alpha)\right]$$

$$= \frac{1}{4\sin 3\alpha}\left[3\sin\alpha + 6\sin(\pi+\alpha)\cos\frac{\pi}{3} - 3\sin 3\alpha\right]$$

$$= \frac{1}{4\sin 3\alpha}[3\sin\alpha - 3\sin\alpha - 3\sin 3\alpha] = -\frac{3}{4}$$

28. (a) : $\frac{\sin\alpha + \sin\beta}{2}$

$$= \frac{2\sin\left(\frac{\alpha+\beta}{2}\right).\cos\left(\frac{\alpha-\beta}{2}\right)}{2} \le \sin\left(\frac{\alpha+\beta}{2}\right)$$

29. (a) : Given that $x, \frac{\pi}{4}, y$ are in A.P

$$\Rightarrow \quad x + y = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan(x+y) = \tan\frac{\pi}{2} \Rightarrow \tan x \tan y = 1$$

$$\tan x \tan\frac{\pi}{4} \tan y = 1$$

30. (d) : $\cot\theta + \tan\theta = x \Rightarrow \sin\theta \ \cos\theta = \frac{1}{x}$.
Also, $\sec\theta - \cos\theta = y \Rightarrow \sin\theta \tan\theta = y$.
And, $(x^2y)^{2/3} - (xy^2)^{2/3}$

$$= \left(\frac{1}{\sin^2\theta\cdot\cos^2\theta}\frac{\sin^2\theta}{\cos\theta}\right)^{2/3} - \left(\frac{1}{\sin\theta\cos\theta}.\frac{\sin^4\theta}{\cos^2\theta}\right)^{2/3}$$

$= \sec^2\theta - \tan^2\theta = 1$.

31. (b) : $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

$$\Rightarrow \quad 3\tan\frac{A}{2}\tan\frac{C}{2} = 1$$

$$\left(\because \tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2} \text{ are in H.P.}\right)$$

$$\Rightarrow \quad \cot\frac{A}{2}\cot\frac{C}{2} = 3$$

Using A.M. $\ge$ G. M. in equality, we have

$$\Rightarrow \quad \frac{\cot\frac{A}{2} + \cot\frac{C}{2}}{2} \ge \sqrt{\cot\frac{A}{2}\cot\frac{C}{2}} \quad \Rightarrow \quad \cot\frac{B}{2} \ge \sqrt{3}$$

Hence, the minimum value of $\cot\frac{B}{2} = \sqrt{3}$

32. (c) : We have,
$(m+2)\tan\theta + (2m-1) = (2m+1)\sec\theta$

$\Rightarrow$ $(m+2)^2\tan^2\theta + 2(m+2)(2m-1)\tan\theta + (2m-1)^2 = (2m+1)^2(1+\tan^2\theta)$

$\Rightarrow$ $(3\tan\theta - 4)[(1-m^2)\tan\theta + 2m] = 0$

$$\Rightarrow \quad \tan\theta = \frac{4}{3} \text{ or } \frac{2m}{m^2-1}$$

33. (a) : L.H.S. $= \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$

$= 1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= 2[1 - \cos(\alpha - \beta)]$

$$= 2\left[1 - \left(1 - 2\sin^2\left(\frac{\alpha-\beta}{2}\right)\right)\right]$$

$$= 4\sin^2\left(\frac{\alpha-\beta}{2}\right) = k\sin^2\left(\frac{\alpha-\beta}{2}\right) \Rightarrow k = 4.$$

34. (d) : We have, $x = \frac{1-\sin\phi}{\cos\phi}$, $y = \frac{1+\cos\phi}{\sin\phi}$

$$xy + 1 = \frac{1-\sin\phi+\cos\phi}{\cos\phi\cdot\sin\phi} = -(x-y).$$

Hence, $xy + 1 + x - y = 0$

$$\therefore \quad x = \frac{y-1}{y+1}, \quad y = \frac{1+x}{1-x}.$$

35. (a) : The given expression is
$\log(\cot 1° \cot 2° \cot 3°... \cot 89°)$
$= \log(\cot 1°\cot 2°....\tan 2°\tan 1°) = \log 1 = 0$.

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MATHS MUSING

SOLUTION SET-187

1. **(b) :** Let $z = x + iy$

$$\text{Im}\left[\left(\frac{ix-y-2}{x+i(y-1)}\right)\left(\frac{x-i(y-1)}{x-i(y-1)}\right)\right]+1=0$$

$$\Rightarrow \frac{(y-1)(y+2)+x^2}{x^2+(y-1)^2}+1=0$$

$\Rightarrow 2x^2 + 2y^2 - y - 1 = 0 \Rightarrow x^2 + y^2 - (1/2)y - (1/2) = 0$

$\therefore$ Centre of circle is $\left(0,\frac{1}{4}\right)$.

$$\therefore \text{ Radius } = \sqrt{0+\left(\frac{1}{4}\right)^2+\frac{1}{2}} = \sqrt{\frac{1}{16}+\frac{1}{2}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

2. **(c) :** $T_{r+1} = {}^{6561}C_r\,(7^{1/3})^{6561-r}\cdot(11^{1/9})^r$

$$= {}^{6561}C_r \cdot 7^{2187-\frac{r}{3}} \cdot 11^{r/9}$$

T_{r+1} is rational, if $\frac{r}{9}$ and $\frac{r}{3}$ are integers.

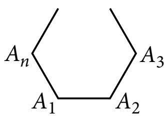
$\therefore$ r is a multiple of 9

$\Rightarrow 0 \le \frac{r}{9} \le 729$ $\qquad (\because\ 0 \le r \le 6561)$

$\therefore \frac{r}{9} = 0, 1, 2, 3, \ldots, 729$

$\therefore$ Total terms = 730

3. **(d) :** Total outcomes = ${}^nC_3$.



Favourable ways = n

$$\therefore \text{ Required probability} = \frac{n}{{}^nC_3} = \frac{6}{(n-1)(n-2)}$$

4. **(a) :** $A = \frac{2}{\sqrt{3}}e^{i\pi/2}, B = \frac{2}{\sqrt{3}}e^{-i\pi/6}, C = \frac{2}{\sqrt{3}}e^{-i5\pi/6}$

$$A = \frac{2}{\sqrt{3}}i, B = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}-i}{2}\right), C = \frac{2}{\sqrt{3}}\left(\frac{-\sqrt{3}-i}{2}\right),$$

$|A - B| = 2, |B - C| = 2, |C - A| = 2$

$\therefore$ ΔABC is equilateral.

5. **(c) :** We have, $(x - 1)(x^2 - 2)(x^3 - 3)(x^4 - 4)\ldots(x^{20} - 20)$ ...(i)

$$= x^{210}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right)\ldots\ldots\left(1-\frac{20}{x^{20}}\right).$$

Consider integer partitions with distinct part of $210 - 203 = 7$, which are given by 7, $6 + 1$, $5 + 2$, $4 + 3$, $4 + 2 + 1$

$\therefore$ Coefficient is given by $-7 + 1\cdot6 + 2\cdot5 + 3\cdot4 - 1\cdot2\cdot4 = 13$.

6. **(a, d) :** By hypothesis, $q = \frac{2pr}{p+r}$

$\Rightarrow \frac{q}{2} = \frac{pr}{p+r} = k$ (say)

$\Rightarrow q = 2k, pr = k(p + r)$

Also, p^2, q^2, r^2 are in A.P.

$2q^2 = p^2 + r^2 = (p + r)^2 - 2pr$

$8k^2 = (p + r)^2 - 2k(p + r)$

or $(p + r)^2 - 2(p + r)k - 8k^2 = 0$

$\Rightarrow p + r = 4k, -2k$

If $p + r = 4k \Rightarrow pr = 4k^2$

and $(p - r)^2 = (p + r)^2 - 4pr = 0 \Rightarrow p = r$

This is against the hypothesis.

$\therefore p + r = -2k;\ pr = -2k^2$

Now, $(p - r)^2 = 12k^2 \Rightarrow p - r = \pm 2\sqrt{3}k$

Combine it with $p + r = -2k$ to get

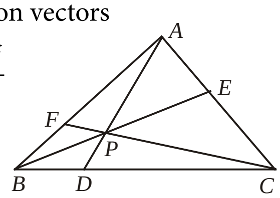
$p = (-1 \pm \sqrt{3})k$ and $r = (-1 \mp \sqrt{3})k$

$\therefore p : q : r :: -1 \pm \sqrt{3} : 2 : -1 \mp \sqrt{3}$

or $p : q : r :: 1 \mp \sqrt{3} : -2 : 1 \pm \sqrt{3}$

7. **(a) :** If A is the origin.

Let $\overrightarrow{AB} = \vec{b}, \overrightarrow{AC} = \vec{c}$ be position vectors



$BD : DC = 1 : 2 \Rightarrow D = \frac{2\vec{b}+\vec{c}}{3}$

$AE : EC = 2 : 3 \Rightarrow E = \frac{2\vec{c}}{5}$

The line AD is $\vec{r} = s\left(\frac{2\vec{b}+\vec{c}}{3}\right)$

The line BE is $\vec{r} = \vec{b} + t\left(\frac{2\vec{c}}{5} - \vec{b}\right)$

They meet at P: $\frac{2s}{3} = 1 - t, \frac{s}{3} = \frac{2t}{5} \Rightarrow s = \frac{2}{3}, t = \frac{5}{9}$

$$\therefore P = \frac{2}{9}(2\vec{b}+\vec{c}) \Rightarrow \frac{AP}{PD} = \frac{\frac{2}{9}}{\frac{1}{3}-\frac{2}{9}} = \frac{2}{1}.$$

Solution Sender of Maths Musing

SET-187

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SET-186

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8. (c) : $\dfrac{BP}{PE}=\dfrac{\frac{2}{9}(2\vec{b}+\vec{c})-\vec{b}}{\frac{2\vec{c}}{5}-\frac{2}{9}(2\vec{b}+\vec{c})}=\dfrac{5(2\vec{c}-5\vec{b})}{8\vec{c}-20\vec{b}}=\dfrac{5}{4}.$

9. (3) : $\lim\limits_{x\to -1}\dfrac{f(x)}{(x+1)^3}=1$

$\Rightarrow\ f(-1)=f'(-1)=f''(-1)=0,\ f'''(-1)=6$

$\therefore\ f(x)=A(x+1)^4+(x+1)^3$

$f'''(x)=24A(x+1)+6$

$f'''(0)=0\Rightarrow 24A+6=0\quad\Rightarrow\ A=-\dfrac{1}{4}$

$\therefore\ f(x)=(x+1)^3-\dfrac{1}{4}(x+1)^4$

$f'(x)=3(x+1)^2-(x+1)^3$

$f''(x)=6(x+1)-3(x+1)^2$

$f'(x)=0\Rightarrow (x+1)^2(2-x)=0$

$\Rightarrow\ x=-1,\ x=2$ and $f''(2)=-9$

$\therefore$ The maximum value of $f(x)$ is $f(2)=27-\dfrac{81}{4}=\dfrac{27}{4}$

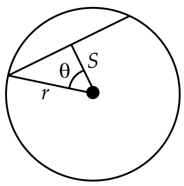
Thus, $\dfrac{9}{4}\lambda=\dfrac{27}{4}\Rightarrow\lambda=3$

10. (a) : (P) → 2, (Q) → 1, (R) → 4, (S) → 3

(P) Let P be the length of the chord, S be the distance of the mid point of the chord from the centre and r be the radius of the given circle, then

$S=r\cos\theta,\ P=2r\sin\theta$

Now $\dfrac{1}{4}(2r)<2r\sin\theta<\dfrac{3}{4}\ 2r$



$\Rightarrow\ \dfrac{1}{4}<\sqrt{1-\cos^2\theta}<\dfrac{3}{4}$

$\Rightarrow\ \dfrac{\sqrt{7}}{4}r<S<\dfrac{\sqrt{15}}{4}r$

Let $R_1=\dfrac{\sqrt{7}}{4}r,\ R_2=\dfrac{\sqrt{15}}{4}r$

$\therefore$ Given condition is satisfied if mid point of the chord lies with in region between concentric circles of radii $\dfrac{\sqrt{7}}{4}r$ and $\dfrac{\sqrt{15}}{4}r$.

$\therefore$ Required probability $=\dfrac{\text{Area of the circular annulus}}{\text{Area of given circle}}$

$=\dfrac{\pi(R_2^{\,2}-R_1^{\,2})}{\pi r^2}=\left(\dfrac{15}{16}-\dfrac{7}{16}\right)=\dfrac{1}{2}$

(Q) 1, 2, 4; 4, 6, 9; 2, 4, 8; 1, 3, 9 are 4 G.P.s

$\therefore$ Required probability $=\dfrac{4}{{}^{10}C_3}=\dfrac{4}{120}=\dfrac{1}{30}$

(R) 1, 4, 7, 10; 2, 5, 8; 3, 6, 9

x^3+y^3 is divisible by $3\Rightarrow x+y$ is divisible by 3.

$\therefore$ x in the first row above and y in the second row or x in the second row and y in the first row or both x and y are in the third row.

$\therefore$ Required probability $=\dfrac{4\times 3+3}{{}^{10}C_2}=\dfrac{15}{45}=\dfrac{1}{3}$

(S) Let SS stand for spade lost and spade drawn, NS stand for non-spade lost and spade drawn.

$\therefore$ Required probability $=P(SS)+P(NS)$

$=\dfrac{1}{4}\cdot\dfrac{12}{51}+\dfrac{3}{4}\cdot\dfrac{13}{51}=\dfrac{1}{4}$

SAMURAI SUDOKU

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique solution.

	9	4					1						3					8	4	
1						4		5				9		6						3
7					2		3						1		3					2
				1		6								7		5				
			7		6										8		7			
		3		8												6		4		
	4		3					1				7					1		5	
5		2						7				1						2		8
	1					3	9		1		6		5	8					9	
								5		8		3								
									5		1									
								6		4		8								
	6					7	3		4		2		1	6					2	
7		4						9				4						7		3
	8		7					2				5					4		9	
		6		2												4		5		
			3		7										2		9			
				5		8								3		1				
3					8		7						2		7					9
9						3		4				6		1						7
	4	5					8						9					2	8	

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

Samurai Sudoku Solution Sender (July 2018)

- Akanksha Das (West Bengal)
- Sandeepa Dhara (West Bengal)
- Chirag Mutha